

# Logarithmic tensor category theory, VIII: Braided tensor category structure on categories of generalized modules for a conformal vertex algebra

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## Abstract

This is the eighth part in a series of papers in which we introduce and develop a natural, general tensor category theory for suitable module categories for a vertex (operator) algebra. In this paper (Part VIII), we construct the braided tensor category structure, using the previously developed results.

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In this paper, Part VIII of a series of eight papers on logarithmic tensor category theory, we construct the braided tensor category structure, using the previously developed results. The sections, equations, theorems and so on are numbered globally in the series of papers rather than within each paper, so that for example equation (a.b) is the b-th labeled equation in Section a, which is contained in the paper indicated as follows: In Part I [HLZ1], which contains Sections 1 and 2, we give a detailed overview of our theory, state our main results and introduce the basic objects that we shall study in this work. We include a brief discussion of some of the recent applications of this theory, and also a discussion of some recent literature. In Part II [HLZ2], which contains Section 3, we develop logarithmic formal calculus and study logarithmic intertwining operators. In Part III [HLZ3], which contains Section 4, we introduce and study intertwining maps and tensor product bifunctors. In Part IV [HLZ4], which contains Sections 5 and 6, we give constructions of the  $P(z)$ - and

$Q(z)$ -tensor product bifunctors using what we call “compatibility conditions” and certain other conditions. In Part V [HLZ5], which contains Sections 7 and 8, we study products and iterates of intertwining maps and of logarithmic intertwining operators and we begin the development of our analytic approach. In Part VI [HLZ6], which contains Sections 9 and 10, we construct the appropriate natural associativity isomorphisms between triple tensor product functors. In Part VII [HLZ7], which contains Section 11, we give sufficient conditions for the existence of the associativity isomorphisms. The present paper, Part VIII, contains Section 12.

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## 12 The braided tensor category structure

In this section, we shall complete the formulations and proofs of our main theorems. We construct a natural braided monoidal category structure on the category  $\mathcal{C}$ . In particular, when  $\mathcal{C}$  is an abelian category, we obtain a natural braided tensor category structure on  $\mathcal{C}$ . The strategy and steps in our construction in this section are essentially the same as those in [HL1], [HL5] and [H2] in the finitely reductive case but, instead of the corresponding constructions and results in [HL2], [HL3], [HL4] and [H1], we of course use all the constructions and results we have obtained in this work except for those in Section 11. The present section is independent of Section 11, which provided a method for verifying the relevant hypotheses.

Our constructions and proofs in this work actually give much more, namely, the vertex-tensor-categorical structure, in the sense of [HL1], relevant for producing the desired braided tensor category structure. We have constructed tensor product bifunctors depending on a nonzero complex number  $z$ , along with associativity isomorphisms between suitable trifunctors constructed from these bifunctors, and in this section, we shall first give natural isomorphisms between certain additional functors constructed from them. These structures, when enhanced by natural isomorphisms constructed from the Virasoro algebra operators, actually give vertex tensor category structure (in the sense of [HL1]). Our construction of braided tensor category structure in this section is simply a byproduct of this vertex tensor categorical-type structure. In particular, we choose the tensor product bifunctor for our braided tensor category structure to be the tensor product bifunctor associated to  $z = 1$  and we construct all the other necessary data from the natural isomorphisms mentioned above. This process of specializing all our tensor product bifunctors to the bifunctor associated with only one particular nonzero complex number “forgets” all of the essential complex-analytic vertex-tensor-categorical structure developed in the present work, except for only the “topological” information, which is what braided tensor category structure exhibits. But even if we are interested only in constructing our braided tensor category structure, we are still forced to construct the vertex-tensor-categorical structure first, because, for instance,

iterated tensor products of triples of elements *are not defined in the braided tensor category structure*.

We now return to the setting and assumptions of Section 10, in particular, Assumption 10.1 and also Remark 10.2, in which we have made a choice  $\boxtimes_{P(z)} = \boxdot'_{P(z)}$  of the tensor product bifunctors. In addition, we also make the following two assumptions, the second of which implies the convergence condition for intertwining maps in  $\mathcal{C}$  (where we take  $W_3 = V$ ,  $M_2 = W_4$ ,  $\mathcal{Y}_3 = Y_{W_4}$  and  $w_{(3)} = \mathbf{1}$ , and we invoke Proposition 7.3; see Definition 7.4):

**Assumption 12.1** *The vertex algebra  $V$  as a  $V$ -module is an object of  $\mathcal{C}$ .*

**Assumption 12.2** *The expansion condition for intertwining maps in  $\mathcal{C}$  holds (see Definition 9.28). Moreover, for objects  $W_1, W_2, W_3, W_4, W_5, M_1$  and  $M_2$  of  $\mathcal{C}$ , logarithmic intertwining operators  $\mathcal{Y}_1, \mathcal{Y}_2$  and  $\mathcal{Y}_3$  of types  $\binom{W_5}{W_1 M_1}, \binom{M_1}{W_2 M_2}$  and  $\binom{M_2}{W_3 W_4}$ , respectively,  $z_1, z_2, z_3 \in \mathbb{C}^\times$  satisfying  $|z_1| > |z_2| > |z_3| > 0$ , and  $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ , the series*

$$\sum_{m,n \in \mathbb{R}} \langle w'_{(5)}, \mathcal{Y}_1(w_{(1)}, z_1) \pi_m(\mathcal{Y}_2(w_{(2)}, z_2) \pi_n(\mathcal{Y}_3(w_{(3)}, z_3) w_{(4)})) \rangle_{W_5} \quad (12.1)$$

*is absolutely convergent and can be analytically extended to a multivalued analytic function on the region given by  $z_1, z_2, z_3 \neq 0, z_1 \neq z_2, z_1 \neq z_3$  and  $z_2 \neq z_3$ , such that for any set of possible singular points with either  $z_1 = 0, z_2 = 0, z_3 = 0, z_1 = \infty, z_2 = \infty, z_3 = \infty, z_1 = z_2, z_1 = z_3$  or  $z_2 = z_3$ , this multivalued analytic function can be expanded near the singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points (as defined in Appendix B of [Kn]; recall Section 11.2).*

**Remark 12.3** By Theorems 11.6 and 11.8 (see also Remark 11.7), when  $A$  and  $\tilde{A}$  are trivial, Assumption 12.2 holds if every object of  $\mathcal{C}$  satisfies the  $C_1$ -cofiniteness condition and the quasi-finite dimensionality condition, or, when  $\mathcal{C}$  is in  $\mathcal{M}_{sg}$ , if every object of  $\mathcal{C}$  is a direct sum of irreducible objects of  $\mathcal{C}$  and there are only finitely many irreducible  $C_1$ -cofinite objects of  $\mathcal{C}$  (up to equivalence).

The main result of this work is Theorem 12.15, which states that if  $V$  is a Möbius or conformal vertex algebra and  $\mathcal{C}$  is a full subcategory of  $\mathcal{M}_{sg}$  or  $\mathcal{GM}_{sg}$  satisfying Assumptions 10.1, 12.1 and 12.2, then the category  $\mathcal{C}$ , equipped with the tensor product bifunctor  $\boxtimes$ , the unit object  $V$ , the braiding isomorphisms  $\mathcal{R}$ , the associativity isomorphisms  $\mathcal{A}$ , and the left and right unit isomorphisms  $l$  and  $r$ , is an additive braided monoidal category. In particular, if  $\mathcal{C}$  is an abelian category, then equipped with the data above, it is a braided tensor category.

## 12.1 More on tensor products of elements

Let  $W_1, W_2$  and  $W_3$  be objects of  $\mathcal{C}$ . In Section 7, using the convergence condition, for  $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$  satisfying  $|z_1| > |z_2| > 0$  and  $|z_3| > |z_4| > 0$ , we have defined the tensor

product elements

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$$

and

$$(w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)} \in \overline{(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3},$$

respectively, for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ . In the proof of the commutativity of the hexagon diagrams below, we shall also need tensor products of elements  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$  in  $\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$  and in  $\overline{(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3}$  when  $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$  satisfy  $z_1 \neq z_2$  and  $z_3 \neq z_4$  but do not necessarily satisfy the inequality  $|z_1| > |z_2| > 0$  or  $|z_3| > |z_4| > 0$ . Here we first define these elements.

Let  $\mathcal{Y}_1 = \mathcal{Y}_{\boxtimes_{P(z_1)}, 0}$ ,  $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes_{P(z_2)}, 0}$ ,  $\mathcal{Y}_3 = \mathcal{Y}_{\boxtimes_{P(z_3)}, 0}$  and  $\mathcal{Y}_4 = \mathcal{Y}_{\boxtimes_{P(z_4)}, 0}$  be intertwining operators of types

$$\begin{aligned} & \begin{pmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ W_1 \quad W_2 \boxtimes_{P(z_2)} W_3 \end{pmatrix}, \\ & \begin{pmatrix} W_2 \boxtimes_{P(z_2)} W_3 \\ W_2 \quad W_3 \end{pmatrix}, \\ & \begin{pmatrix} (W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3 \\ W_1 \boxtimes_{P(z_4)} W_2 \quad W_3 \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} W_1 \boxtimes_{P(z_4)} W_2 \\ W_1 \quad W_2 \end{pmatrix},$$

respectively, corresponding to the intertwining maps  $\boxtimes_{P(z_1)}$ ,  $\boxtimes_{P(z_2)}$ ,  $\boxtimes_{P(z_3)}$  and  $\boxtimes_{P(z_4)}$ , respectively, as in (4.17) and (4.18). Then by Assumption 12.2,

$$\langle w', \mathcal{Y}_1(w_{(1)}, \zeta_1) \mathcal{Y}_2(w_{(2)}, \zeta_2) w_{(3)} \rangle$$

and

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_{(1)}, \zeta_4) w_{(2)}, \zeta_3) w_{(3)} \rangle$$

are absolutely convergent for

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3)',$$

when  $|\zeta_1| > |\zeta_2| > 0$  and when  $|\zeta_3| > |\zeta_4| > 0$ , respectively, and can be analytically extended to multivalued analytic functions in the regions given by  $\zeta_1, \zeta_2 \neq 0$  and  $\zeta_1 \neq \zeta_2$  and by  $\zeta_3, \zeta_4 \neq 0$  and  $\zeta_3 \neq -\zeta_4$ , respectively. If we cut these regions along  $\zeta_1, \zeta_2 \geq \mathbb{R}_+$  and  $\zeta_3, \zeta_4 \in \mathbb{R}_+$ , respectively, we obtain simply-connected regions and we can choose single-valued branches of these multivalued analytic functions. In particular, we have the branches of these two

multivalued analytic functions such that their values at points satisfying  $|\zeta_1| > |\zeta_2| > 0$  and  $|\zeta_3| > |\zeta_4| > 0$  are

$$\langle w', \mathcal{Y}_1(w_{(1)}, \zeta_1) \mathcal{Y}_2(w_{(2)}, \zeta_2) w_{(3)} \rangle$$

and

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_{(1)}, \zeta_4) w_{(2)}, \zeta_3) w_{(3)} \rangle,$$

respectively.

We immediately have:

**Proposition 12.4** *Let  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ . Then for any  $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$  satisfying  $z_1 \neq z_2$  and  $z_3 \neq z_4$ , there exist unique elements*

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}$$

and

$$(w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)} \in \overline{(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3}$$

such that for any

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3)',$$

the numbers

$$\langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle \quad (12.2)$$

and

$$\langle \tilde{w}', (w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)} \rangle \quad (12.3)$$

are the values at  $(\zeta_1, \zeta_2) = (z_1, z_2)$  and  $(\zeta_3, \zeta_4) = (z_3, z_4)$ , respectively, of the branches of the multivalued analytic functions above of  $\zeta_1$  and  $\zeta_2$  and of  $\zeta_3$  and  $\zeta_4$  above, respectively.  $\square$

**Remark 12.5** From the definition of

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$$

and

$$(w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)},$$

we see that when  $|z_1| = |z_2|$  (with  $z_1 \neq z_2$ ) or  $|z_3| = |z_4|$  (with  $z_3 \neq z_4$ ), they are uniquely determined by

$$\begin{aligned} & \langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle \\ &= \lim_{\zeta_1 \rightarrow z_1, \zeta_2 \rightarrow z_2, |\zeta_1| > |\zeta_2| > 0} \langle w', \mathcal{Y}_1(w_{(1)}, \zeta_1) \mathcal{Y}_2(w_{(2)}, \zeta_2) w_{(3)} \rangle \end{aligned}$$

and

$$\begin{aligned} & \langle \tilde{w}', (w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)} \rangle \\ &= \lim_{\zeta_3 \rightarrow z_4, \zeta_4 \rightarrow z_4, |\zeta_3| > |\zeta_4| > 0} \langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_{(1)}, \zeta_4) w_{(2)}) \zeta_3 w_{(3)} \rangle \end{aligned}$$

for

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3)',$$

where the limits take place in the complex plane with a cut along  $\mathbb{R}_+$ .

Propositions 7.16 and 7.18, Corollaries 7.17 and 7.19 and the definitions of tensor products of three elements above immediately give:

**Proposition 12.6** *For any  $z_1, z_2, z_3, z_4 \in \mathbb{C}^\times$  satisfying  $z_1 \neq z_2$  and  $z_3 \neq -z_4$ , the elements of the form*

$$\begin{aligned} &\pi_n(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})), \\ &\pi_n((w_{(1)} \boxtimes_{P(z_4)} w_{(2)}) \boxtimes_{P(z_3)} w_{(3)}) \end{aligned}$$

for  $n \in \mathbb{R}$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$  span

$$\begin{aligned} &W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3), \\ &(W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3, \end{aligned}$$

respectively.  $\square$

Next we discuss tensor products of four elements. These are needed in the proof of the commutativity of the pentagon diagram below.

**Proposition 12.7** *1. Let  $W_1, W_2, W_3, W_4, W_5, M_1$  and  $M_2$  be objects of  $\mathcal{C}$ ,  $z_1, z_2, z_3$  nonzero complex numbers satisfying  $|z_1| > |z_2| > |z_3| > 0$ , and  $I_1, I_2$  and  $I_3$   $P(z_1)$ -,  $P(z_2)$ - and  $P(z_3)$ -intertwining maps of type  $\binom{W_5}{W_1 M_1}$ ,  $\binom{M_1}{W_2 M_2}$  and  $\binom{M_2}{W_3 W_4}$ , respectively. Then for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ , the series*

$$\sum_{m, n \in \mathbb{R}} \langle w'_{(5)}, I_1(w_{(1)} \otimes \pi_m(I_2(w_{(2)} \otimes \pi_n(I_3(w_{(3)} \otimes w_{(4)})))) \rangle_{W_5} \quad (12.4)$$

*is absolutely convergent.*

*2. Let  $W_1, W_2, W_3, W_4, W_5, M_3$  and  $M_4$  be objects of  $\mathcal{C}$ ,  $z_1, z_{23}, z_3$  nonzero complex numbers satisfying  $|z_3| > |z_{23}| > 0$  and  $|z_1| > |z_3| + |z_{23}| > 0$ , and  $I_1, I_2$  and  $I_3$   $P(z_1)$ -,  $P(z_3)$ - and  $P(z_{23})$ -intertwining maps of type  $\binom{W_5}{W_1 M_3}$ ,  $\binom{M_3}{M_4 W_4}$  and  $\binom{M_4}{W_2 W_3}$ , respectively. Then for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ , the series*

$$\sum_{m, n \in \mathbb{R}} \langle w'_{(5)}, I_1(w_{(1)} \otimes \pi_m(I_2(\pi_n(I_3(w_{(2)} \otimes w_{(3)})) \otimes w_{(4)}))) \rangle_{W_5} \quad (12.5)$$

*is absolutely convergent.*

3. Let  $W_1, W_2, W_3, W_4, W_5, M_5$  and  $M_6$  be objects of  $\mathcal{C}$ ,  $z_3, z_{13}, z_{23}$  nonzero complex numbers satisfying  $|z_3| > |z_{13}| > |z_{23}| > 0$ , and  $I_1, I_2$  and  $I_3$   $P(z_3)$ -,  $P(z_{13})$ - and  $P(z_{23})$ -intertwining maps of type  $\binom{W_5}{M_5 W_4}$ ,  $\binom{M_5}{W_1 M_6}$  and  $\binom{M_6}{W_2 W_3}$ , respectively. Then for  $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ , the series

$$\sum_{m,n \in \mathbb{R}} \langle w'_{(5)}, I_1(\pi_m(I_2(w_{(1)} \otimes \pi_n(I_3(w_{(2)} \otimes w_{(3)})))) \otimes w_{(4)} \rangle_{W_5} \quad (12.6)$$

is absolutely convergent.

4. Let  $W_1, W_2, W_3, W_4, W_5, M_7$  and  $M_8$  be objects of  $\mathcal{C}$ ,  $z_3, z_{23}, z_{12}$  nonzero complex numbers satisfying  $|z_{23}| > |z_{12}| > 0$  and  $|z_3| > |z_{23}| + |z_{12}| > 0$ , and  $I_1, I_2$  and  $I_3$   $P(z_3)$ -,  $P(z_{23})$ - and  $P(z_{12})$ -intertwining maps of type  $\binom{W_5}{M_7 W_4}$ ,  $\binom{M_7}{M_8 W_3}$  and  $\binom{M_8}{W_1 W_2}$ , respectively. Then for  $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ , the series

$$\sum_{m,n \in \mathbb{R}} \langle w'_{(5)}, I_1(\pi_m(I_2(\pi_n(I_3(w_{(1)} \otimes w_{(2)})) \otimes w_{(3)})) \otimes w_{(4)} \rangle_{W_5} \quad (12.7)$$

is absolutely convergent.

5. Let  $W_1, W_2, W_3, W_4, W_5, M_9$  and  $M_{10}$  be objects of  $\mathcal{C}$ ,  $z_{12}, z_2, z_3$  nonzero complex numbers satisfying  $|z_2| > |z_{12}| + |z_3| > 0$ , and  $I_1, I_2$  and  $I_3$   $P(z_2)$ -,  $P(z_{12})$ - and  $P(z_3)$ -intertwining maps of type  $\binom{W_5}{M_9 M_{10}}$ ,  $\binom{M_9}{W_1 W_2}$  and  $\binom{M_{10}}{W_3 W_4}$ , respectively. Then for  $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ , the series

$$\sum_{m,n \in \mathbb{R}} \langle w'_{(5)}, I_1(\pi_m(I_2(w_{(1)} \otimes w_{(2)})) \otimes \pi_n(I_2(w_{(3)} \otimes w_{(4)}))) \rangle_{W_5} \quad (12.8)$$

is absolutely convergent.

*Proof* The absolute convergence of (12.4) follows immediately from Assumption 12.2 and Proposition 4.8.

To prove the absolute convergence of (12.5), let  $\mathcal{Y}_1, \mathcal{Y}_2$  and  $\mathcal{Y}_3$  be the intertwining operators corresponding to  $I_1, I_2$  and  $I_3$ , using  $p = 0$  as usual, that is,  $\mathcal{Y}_1 = \mathcal{Y}_{I_1,0}, \mathcal{Y}_2 = \mathcal{Y}_{I_2,0}$  and  $\mathcal{Y}_3 = \mathcal{Y}_{I_3,0}$ . We would like to prove that for  $w_{(1)} \in W_1, w_{(2)} \in W_2, w_{(3)} \in W_3, w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ ,

$$\sum_{m,n \in \mathbb{R}} \langle w'_{(5)}, \mathcal{Y}_1(w_{(1)}, z_1) \pi_m(\mathcal{Y}_2(\pi_n(\mathcal{Y}_3(w_{(2)}, z_{23})w_{(3)}), z_3)w_{(4)})) \rangle_{W_5} \quad (12.9)$$

is absolutely convergent when  $|z_3| > |z_{23}| > 0$  and  $|z_1| > |z_3| + |z_{23}| > 0$ . By the  $L(0)$ -conjugation property for intertwining operators, (12.9) is equal to

$$\begin{aligned} & \sum_{m,n \in \mathbb{R}} \langle e^{(\log z_3)L'(0)} w'_{(5)}, \mathcal{Y}_1(e^{-(\log z_3)L(0)} w_{(1)}, z_1 z_3^{-1}) \cdot \\ & \quad \cdot \pi_m(\mathcal{Y}_2(\pi_n(\mathcal{Y}_3(e^{-(\log z_3)L(0)} w_{(2)}, z_{23} z_3^{-1}) e^{-(\log z_3)L(0)} w_{(3)}), 1) e^{-(\log z_3)L(0)} w_{(4)}) \rangle_{W_5}. \end{aligned} \quad (12.10)$$

Since (12.9) is equal to (12.10), we see that proving that for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ , (12.9) is absolutely convergent when  $|z_3| > |z_{23}| > 0$  and  $|z_1| > |z_3| + |z_{23}| > 0$  is equivalent to proving that for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w_{(4)} \in W_4$  and  $w'_{(5)} \in W'_5$ ,

$$\sum_{m,n \in \mathbb{R}} \langle w'_{(5)}, \mathcal{Y}_1(w_{(1)}, \zeta_1) \pi_m(\mathcal{Y}_2(\pi_n(\mathcal{Y}_3(w_{(2)}, \zeta_{23})w_{(3)}), 1)w_{(4)})) \rangle_{W_5} \quad (12.11)$$

is absolutely convergent when  $1 > |\zeta_{23}| > 0$  and  $|\zeta_1| > 1 + |\zeta_{23}| > 0$ .

Using Corollary 9.30 (the associativity of intertwining operators), we know that there exist an object  $M$  of  $\mathcal{C}$  and intertwining operators  $\mathcal{Y}_4$  and  $\mathcal{Y}_5$  of types  $\binom{M_3}{W_2 M}$  and  $\binom{M}{W_3 W_4}$ , respectively, such that for  $w' \in M'_3$ , when  $1 > |\zeta_{23}| > 0$ , the series

$$\sum_{n \in \mathbb{R}} \langle w', \mathcal{Y}_2(\pi_n(\mathcal{Y}_3(w_{(2)}, \zeta_{23})w_{(3)}), 1)w_{(4)} \rangle_{M_3} = \langle w', \mathcal{Y}_2(\mathcal{Y}_3(w_{(2)}, \zeta_{23})w_{(3)}), 1)w_{(4)} \rangle_{M_3}$$

is absolutely convergent and, when  $1 > |\zeta_{23}| > 0$  and  $|\zeta_{23} + 1| > 1$ , its sum is equal to

$$\langle w', \mathcal{Y}_4(w_{(2)}, \zeta_{23} + 1) \mathcal{Y}_5(w_{(3)}, 1)w_{(4)} \rangle_{M_3}.$$

By Assumption 12.2, we know that

$$\sum_{m,n \in \mathbb{R}} \langle w'_{(5)}, \mathcal{Y}_1(w_{(1)}, \zeta_1) \pi_m(\mathcal{Y}_4(w_{(2)}, \zeta_{23} + 1) \pi_n(\mathcal{Y}_5(w_{(3)}, 1)w_{(4)})) \rangle_{M_3} \quad (12.12)$$

is absolutely convergent and can be analytically extended to a multivalued analytic function  $f(\zeta_1, \zeta_{23})$  on the region given by  $\zeta_1, \zeta_{23} + 1 \neq 0$ ,  $\zeta_1 \neq \zeta_{23} + 1$ ,  $\zeta_1 \neq 1$  and  $\zeta_{23} + 1 \neq 1$  such that for any set of possible singular points with either  $\zeta_1 = 0$ ,  $\zeta_{23} + 1 = 0$ ,  $\zeta_1 = \zeta_{23} + 1$ ,  $\zeta_1 = 1$ ,  $\zeta_{23} = 0$ ,  $\zeta_1 = \infty$  or  $\zeta_{23} + 1 = \infty$ , this multivalued analytic function can be expanded near its singularity as a series having the same form as the expansion near the singular points of a solution of a system of differential equations with regular singular points (again, as defined in Appendix B of [Kn]; recall Section 11.2). Since  $(\zeta_1, \zeta_{23}) = (\infty, 0)$  gives such a set of possible singular points of  $f(\zeta_1, \zeta_{23})$ , and since for fixed  $\zeta_1, \zeta_{23}$  satisfying  $|\zeta_1| > |\zeta_{23}| + 1$  and  $0 < |\zeta_{23}| < 1$ , there exists a positive real number  $r < 1$  such that  $|\zeta_1| > r + 1$  and  $0 < |\zeta_{23}| < r$ ,  $f(\zeta_1, \zeta_{23})$  can be expanded as a series in powers of  $\zeta_1$  and  $\zeta_{23}$  and in nonnegative integral powers of  $\log \zeta_1$  and  $\log \zeta_{23}$  when  $|\zeta_1| > |\zeta_{23}| + 1$  and  $0 < |\zeta_{23}| < 1$ . Thus we see that (12.11) is in fact one value of this expansion of  $f(\zeta_1, \zeta_{23})$ , proving that (12.11) is absolutely convergent when  $|\zeta_1| > |\zeta_{23}| + 1$  and  $0 < |\zeta_{23}| < 1$ .

The proofs of the absolute convergence of (12.6)–(12.8) are similar.  $\square$

The special case that the  $P(\cdot)$ -intertwining maps considered are  $\boxtimes_{P(\cdot)}$  gives the following:

**Corollary 12.8** *Let  $W_1, W_2, W_3, W_4$  be objects of  $\mathcal{C}$  and  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w_{(4)} \in W_4$ . Then we have:*



1. For  $z_1, z_2, z_3 \in \mathbb{C}^\times$  satisfying  $|z_1| > |z_2| > |z_3| > 0$  and

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)))' = W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)),$$

the series

$$\sum_{m,n \in \mathbb{R}} \langle w', w_{(1)} \boxtimes_{P(z_1)} \pi_m(w_{(2)} \boxtimes_{P(z_2)} \pi_n(w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \rangle \quad (12.13)$$

is absolutely convergent.

2. For  $z_1, z_{23}, z_3 \in \mathbb{C}^\times$  satisfying  $|z_3| > |z_{23}| > 0$  and  $|z_1| > |z_3| + |z_{23}| > 0$  and

$$w' \in (W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4))' = W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4),$$

the series

$$\sum_{m,n \in \mathbb{R}} \langle w', w_{(1)} \boxtimes_{P(z_1)} \pi_m(\pi_n(w_{(2)} \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)}) \rangle \quad (12.14)$$

is absolutely convergent.

3. For  $z_3, z_{13}, z_{23} \in \mathbb{C}^\times$  satisfying  $|z_3| > |z_{13}| > |z_{23}| > 0$  and

$$w' \in ((W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4)' = (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4,$$

the series

$$\sum_{m,n \in \mathbb{R}} \langle w', \pi_m(w_{(1)} \boxtimes_{P(z_{13})} \pi_n(w_{(2)} \boxtimes_{P(z_{23})} w_{(3)})) \boxtimes_{P(z_3)} w_{(4)} \rangle \quad (12.15)$$

is absolutely convergent.

4. For  $z_3, z_{23}, z_{12} \in \mathbb{C}^\times$  satisfying  $|z_{23}| > |z_{12}| > 0$  and  $|z_3| > |z_{23}| + |z_{12}| > 0$  and

$$w' \in (((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4)' = ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4,$$

the series

$$\sum_{m,n \in \mathbb{R}} \langle w', \pi_m(\pi_n(w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)}) \rangle \quad (12.16)$$

is absolutely convergent.

5. For  $z_{12}, z_2, z_3 \in \mathbb{C}^\times$  satisfying  $|z_2| > |z_{12}| + |z_3| > 0$  and

$$w' \in ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))' = (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4),$$

the series

$$\sum_{m,n \in \mathbb{R}} \langle w', \pi_m(w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} \pi_n(w_{(3)} \boxtimes_{P(z_3)} w_{(4)}) \rangle \quad (12.17)$$

is absolutely convergent.  $\square$

Note that the sums of (12.13), (12.14), (12.15), (12.16), (12.17) define elements of

$$\overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))}, \quad (12.18)$$

$$\overline{W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_1)} W_4)}, \quad (12.19)$$

$$\overline{(W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4}, \quad (12.20)$$

$$\overline{((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4}, \quad (12.21)$$

$$\overline{(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)}, \quad (12.22)$$

respectively, for  $z_1, z_2, z_3, z_{12}, z_{13}, z_{23}$  satisfying the corresponding inequalities. We shall denote these five elements by

$$\begin{aligned} & w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)})), \\ & w_{(1)} \boxtimes_{P(z_1)} ((w_{(2)} \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_1)} w_{(4)}), \\ & (w_{(1)} \boxtimes_{P(z_{13})} (w_{(2)} \boxtimes_{P(z_{23})} w_{(3)})) \boxtimes_{P(z_3)} w_{(4)}, \\ & ((w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)}, \\ & (w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}), \end{aligned}$$

respectively.

**Proposition 12.9** *The elements of the form*

$$\begin{aligned} & \pi_n(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}))), \\ & \pi_n(w_{(1)} \boxtimes_{P(z_1)} ((w_{(2)} \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_1)} w_{(4)})), \\ & \pi_n((w_{(1)} \boxtimes_{P(z_{13})} (w_{(2)} \boxtimes_{P(z_{23})} w_{(3)})) \boxtimes_{P(z_3)} w_{(4)}), \\ & \pi_n(((w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)}), \\ & \pi_n((w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \end{aligned}$$

for  $n \in \mathbb{R}$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w_{(4)} \in W_4$  span

$$\begin{aligned} & W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)), \\ & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_1)} W_4), \\ & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4, \\ & ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4, \\ & (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4), \end{aligned}$$

respectively.

*Proof* The proof is essentially the same as those of Corollaries 7.17 and 7.19.  $\square$

**Proposition 12.10** *Let  $W_1, W_2, W_3, W_4$  be objects of  $\mathcal{C}$ ,  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$ ,  $w_{(4)} \in W_4$  and  $z_1, z_2, z_3 \in \mathbb{C}^\times$  such that  $z_{12} = z_1 - z_2 \neq 0$ ,  $z_{13} = z_1 - z_3 \neq 0$ , and  $z_{23} = z_2 - z_3 \neq 0$ . Then:*

1. When  $|z_1| > |z_2| > |z_{12}| + |z_3| > 0$ , we have

$$\begin{aligned} & \overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}} (w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}))) \\ &= (w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}), \end{aligned} \quad (12.23)$$

where  $\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}}$  is the natural extension of  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}$  to (12.18).

2. When  $|z_2| > |z_{12}| + |z_3| > 0$ ,  $|z_3| > |z_{12}| + |z_{23}| > 0$  and  $|z_{23}| > |z_{12}| > 0$ , we have

$$\begin{aligned} & \overline{\mathcal{A}_{P(z_2), P(z_3)}^{P(z_{23}), P(z_3)}} ((w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \\ &= ((w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)}, \end{aligned} \quad (12.24)$$

where  $\overline{\mathcal{A}_{P(z_2), P(z_3)}^{P(z_{23}), P(z_3)}}$  is the natural extension of  $\mathcal{A}_{P(z_2), P(z_3)}^{P(z_{23}), P(z_3)}$  to (12.22).

3. When  $|z_1| > |z_2| > |z_3| > |z_{23}| > 0$ ,  $|z_1| > |z_3| + |z_{23}| > 0$ , we have

$$\begin{aligned} & \overline{1_{W_1} \boxtimes_{P(z_1)} \mathcal{A}_{P(z_2), P(z_3)}^{P(z_{23}), P(z_3)}} (w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}))) \\ &= w_{(1)} \boxtimes_{P(z_1)} ((w_{(2)} \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)}), \end{aligned} \quad (12.25)$$

where  $\overline{1_{W_1} \boxtimes_{P(z_1)} \mathcal{A}_{P(z_2), P(z_3)}^{P(z_{23}), P(z_3)}}$  is the natural extension of  $1_{W_1} \boxtimes_{P(z_1)} \mathcal{A}_{P(z_2), P(z_3)}^{P(z_{23}), P(z_3)}$  to (12.18).

4. When  $|z_3| > |z_{13}| > |z_{23}| > 0$  and  $|z_1| > |z_3| + |z_{23}| > 0$ , we have

$$\begin{aligned} & \overline{\mathcal{A}_{P(z_1), P(z_3)}^{P(z_{13}), P(z_3)}} (w_{(1)} \boxtimes_{P(z_1)} ((w_{(2)} \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)})) \\ &= (w_{(1)} \boxtimes_{P(z_{13})} (w_{(2)} \boxtimes_{P(z_{23})} w_{(3)})) \boxtimes_{P(z_3)} w_{(4)}, \end{aligned} \quad (12.26)$$

where  $\overline{\mathcal{A}_{P(z_1), P(z_3)}^{P(z_{13}), P(z_3)}}$  is the natural extension of  $\mathcal{A}_{P(z_1), P(z_3)}^{P(z_{13}), P(z_3)}$  to (12.19).

5. When  $|z_3| > |z_{13}| > |z_{23}| > |z_{12}| > 0$  and  $|z_3| > |z_{12}| + |z_{23}| > 0$ , we have

$$\begin{aligned} & \overline{\mathcal{A}_{P(z_{13}), P(z_{23})}^{P(z_{12}), P(z_{23})} \boxtimes_{P(z_3)} 1_{W_4}} ((w_{(1)} \boxtimes_{P(z_{13})} (w_{(2)} \boxtimes_{P(z_{23})} w_{(3)})) \boxtimes_{P(z_3)} w_{(4)}) \\ &= ((w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)}, \end{aligned} \quad (12.27)$$

where  $\overline{\mathcal{A}_{P(z_{13}), P(z_{23})}^{P(z_{12}), P(z_{23})} \boxtimes_{P(z_3)} 1_{W_4}}$  is the natural extension of  $\mathcal{A}_{P(z_{13}), P(z_{23})}^{P(z_{12}), P(z_{23})} \boxtimes_{P(z_3)} 1_{W_4}$  to (12.20).

*Proof* To prove (12.23), we note that when  $|z_1| > |z_2| > |z_{12}| + |z_3| > 0$ , by Corollary 12.8, for

$$w' \in (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4),$$

we have the absolutely convergent series

$$\begin{aligned} & \langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \rangle \\ &= \sum_{m \in \mathbb{R}} \langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} \pi_m(w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \rangle \end{aligned}$$

and

$$\begin{aligned} & \langle w', (w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}) \rangle \\ &= \sum_{m \in \mathbb{R}} \langle w', (w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} \pi_m(w_{(3)} \boxtimes_{P(z_3)} w_{(4)}) \rangle. \end{aligned}$$

Using the map  $\left(\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}\right)'$ , we have

$$\begin{aligned} & \left\langle w', \overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}))) \right\rangle \\ &= \left\langle \left(\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}\right)'(w'), w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \right\rangle \\ &= \sum_{m \in \mathbb{R}} \left\langle \left(\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}\right)'(w'), w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} \pi_m(w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \right\rangle \\ &= \sum_{m \in \mathbb{R}} \left\langle w', \overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} \pi_m(w_{(3)} \boxtimes_{P(z_3)} w_{(4)}))) \right\rangle \\ &= \sum_{m \in \mathbb{R}} \langle w', (w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} \pi_m(w_{(3)} \boxtimes_{P(z_3)} w_{(4)}) \rangle \\ &= \langle w', (w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}) \rangle. \end{aligned}$$

Since  $w'$  is arbitrary, we obtain (12.23).

The equalities (12.24)–(12.27) are proved similarly.  $\square$

## 12.2 The data of the braided monoidal category structure

We choose the tensor product bifunctor of the braided tensor category that we are constructing to be the bifunctor  $\boxtimes_{P(1)}$ , and we shall write it simply as:

$$\boxtimes = \boxtimes_{P(1)}.$$

We take the unit object to be  $V$ . For any  $z \in \mathbb{C}^\times$  and any object  $(W, Y_W)$  of  $\mathcal{C}$ , we take the *left  $P(z)$ -unit isomorphism*

$$l_{z;W} : V \boxtimes_{P(z)} W \rightarrow W$$

to be the unique module map from  $V \boxtimes_{P(z)} W$  to  $W$  such that

$$\overline{l_{z;W}} \circ \boxtimes_{P(z)} = I_{Y_W, 0},$$

where  $I_{Y_W,0} = I_{Y_W,p}$  for  $p \in \mathbb{Z}$  is the unique  $P(z)$ -intertwining map associated to the intertwining operator  $Y_W$  of type  $\binom{W}{VW}$ ; the existence and uniqueness of  $l_{z;W}$  are guaranteed by the universal property of the  $P(z)$ -tensor product  $\boxtimes_{P(z)}$ . It is characterized by

$$l_{z;W}(\mathbf{1} \boxtimes_{P(z)} w) = w$$

for  $w \in W$ . (Here we do not need the natural extension  $\overline{l_{z;W}}$  because  $\mathbf{1} \boxtimes_{P(z)} w \in V \boxtimes_{P(z)} W$ .) The isomorphisms  $l_{W;z}$  for  $W \in \text{ob } \mathcal{C}$  give a natural isomorphism  $l_z$  from the functor  $V \boxtimes_{P(z)} \cdot$  to the identity functor  $1_{\mathcal{C}}$  of  $\mathcal{C}$ ; the naturality of  $l_z$  follows immediately from the characterization of  $l_{z;W}$  and (10.1). The *right  $P(z)$ -unit isomorphism*

$$r_{z;W} : W \boxtimes_{P(z)} V \rightarrow W$$

is the unique module map from  $W \boxtimes_{P(z)} V$  to  $W$  such that

$$\overline{r_{z;W}} \circ \boxtimes_{P(z)} = I_{\Omega_0(Y_W),0},$$

where  $I_{\Omega_0(Y_W),0} = I_{\Omega_0(Y_W),p}$  for  $p \in \mathbb{Z}$  is the unique  $P(z)$ -intertwining map associated to the intertwining operator  $\Omega_0(Y_W)$  of type  $\binom{W}{WV}$  (recall (3.77)). It is characterized by

$$\overline{r_{z;W}}(w \boxtimes_{P(z)} \mathbf{1}) = e^{zL(-1)}w$$

for  $w \in W$ . The isomorphisms  $r_{z;W}$  for  $W \in \text{ob } \mathcal{C}$  give a natural isomorphism  $r_z$  from the functor  $\cdot \boxtimes_{P(z)} V$  to the identity functor  $1_{\mathcal{C}}$ . In particular, we have the left unit isomorphism

$$l = l_1 : V \boxtimes \cdot \rightarrow 1_{\mathcal{C}}$$

and the right unit isomorphism

$$r = r_1 : \cdot \boxtimes V \rightarrow 1_{\mathcal{C}}.$$

To give the braiding and associativity isomorphisms, we need “parallel transport isomorphisms” between  $P(z)$ -tensor products with different  $z$ . Let  $W_1$  and  $W_2$  be objects of  $\mathcal{C}$ , let  $z_1, z_2 \in \mathbb{C}^\times$ , and let  $\gamma$  be a path in  $\mathbb{C}^\times$  from  $z_1$  to  $z_2$ . Let  $\mathcal{Y}$  be the logarithmic intertwining operator associated to the  $P(z_2)$ -tensor product  $W_1 \boxtimes_{P(z_2)} W_2$  and  $l(z_1)$  the value of the logarithm of  $z_1$  determined uniquely by  $\log z_2$  and the path  $\gamma$ . Then we have a  $P(z_1)$ -intertwining map  $I$  defined by

$$I(w_{(1)} \otimes w_{(2)}) = \mathcal{Y}(w_{(1)}, e^{l(z_1)})w_{(2)}$$

for  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . The *parallel transport isomorphism*

$$\mathcal{T}_\gamma = \mathcal{T}_{\gamma;W_1,W_2} : W_1 \boxtimes_{P(z_1)} W_2 \rightarrow W_1 \boxtimes_{P(z_2)} W_2$$

associated to  $\gamma$  is defined to be the unique module map such that

$$I = \overline{\mathcal{T}_\gamma} \circ \boxtimes_{P(z_1)},$$

where  $\overline{\mathcal{T}}_\gamma$  is the natural extension of  $\mathcal{T}_\gamma$  to the formal completion  $\overline{W_1 \boxtimes_{P(z_1)} W_2}$  of  $W_1 \boxtimes_{P(z_1)} W_2$ . As in the definition of the left  $P(z)$ -unit isomorphism, the existence and uniqueness are guaranteed by the universal property of the  $P(z_1)$ -tensor product  $\boxtimes_{P(z_1)}$ . The parallel transport isomorphism  $\mathcal{T}_\gamma$  is characterized by

$$\overline{\mathcal{T}}_\gamma(w_{(1)} \boxtimes_{P(z_1)} w_{(2)}) = \mathcal{Y}(w_{(1)}, x)w_{(2)}|_{\log x=l(z_1), x^n=e^{nl(z_1)}, n \in \mathbb{R}}$$

for  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . These isomorphisms give a natural isomorphism, denoted using the same notation  $\mathcal{T}_\gamma$ , from  $\boxtimes_{P(z_1)}$  to  $\boxtimes_{P(z_2)}$ ; the naturality of  $\mathcal{T}_\gamma$  follows immediately from this characterization and (10.1). Since the intertwining map  $I$  depends only on the homotopy class of  $\gamma$ , from the definition we see that the parallel transport isomorphism also depends only on the homotopy class of  $\gamma$ .

For  $z \in \mathbb{C}^\times$ , let  $I$  be the  $P(z)$ -intertwining map of type  $\binom{W_2 \boxtimes_{P(-z)} W_1}{W_1 \ W_2}$  defined by

$$I(w_1 \otimes w_2) = e^{zL(-1)}(w_{(2)} \boxtimes_{P(-z)} w_{(1)})$$

for  $w_1 \in W_1$  and  $w_2 \in W_2$ . We define a *commutativity isomorphism between the  $P(z)$ - and  $P(-z)$ -tensor products* to be the unique module map

$$\mathcal{R}_{P(z)} = \mathcal{R}_{P(z); W_1, W_2} : W_1 \boxtimes_{P(z)} W_2 \rightarrow W_2 \boxtimes_{P(-z)} W_1$$

such that

$$I = \overline{\mathcal{R}_{P(z)}} \circ \boxtimes_{P(z)}.$$

The existence and uniqueness of  $\mathcal{R}_{P(z)}$  are guaranteed by the universal property of the  $P(z)$ -tensor product. By definition, the commutativity isomorphism  $\mathcal{R}_{P(z)}$  is characterized by

$$\overline{\mathcal{R}_{P(z)}}(w_{(1)} \boxtimes_{P(z)} w_{(2)}) = e^{zL(-1)}(w_{(2)} \boxtimes_{P(-z)} w_{(1)})$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ . These isomorphisms give a natural isomorphism, denoted using the same notation  $\mathcal{R}_{P(z)}$ ,

$$\mathcal{R}_{P(z)} : \boxtimes_{P(z)} \rightarrow \boxtimes_{P(-z)} \circ \sigma_{12},$$

where  $\sigma_{12}$  is the permutation functor on  $\mathcal{C} \times \mathcal{C}$ ; the naturality of  $\mathcal{R}_{P(z)}$  follows immediately from this characterization and (10.1).

Let  $\gamma_1^-$  be a (“counterclockwise”) path from  $-1$  to  $1$  in the closed upper half plane with  $0$  deleted,  $\mathcal{T}_{\gamma_1^-}$  the corresponding parallel transport isomorphism. We define the *braiding isomorphism*

$$\mathcal{R} = \mathcal{R}_{W_1, W_2} : W_1 \boxtimes W_2 \rightarrow W_2 \boxtimes W_1$$

for our braided tensor category to be

$$\mathcal{R} = \mathcal{T}_{\gamma_1^-} \circ \mathcal{R}_{P(1)}.$$

This braiding isomorphism  $\mathcal{R}$  can also be defined directly as follows: Let  $I$  be the  $P(1)$ -intertwining map of type  $\binom{W_2 \boxtimes_{P(1)} W_1}{W_1 \ W_2}$  defined by

$$I(w_1 \otimes w_2) = e^{L(-1)}\overline{\mathcal{T}_{\gamma_1^-}}(w_{(2)} \boxtimes_{P(-1)} w_{(1)})$$

for  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . Then  $\mathcal{R}$  is the unique module map

$$\mathcal{R} : W_1 \boxtimes W_2 \rightarrow W_2 \boxtimes W_1$$

such that

$$I = \overline{\mathcal{R}} \circ \boxtimes.$$

It is characterized by

$$\overline{\mathcal{R}}(w_{(1)} \boxtimes_{P(1)} w_{(2)}) = e^{L(-1)} \overline{\mathcal{T}}_{\gamma_1}^-(w_{(2)} \boxtimes_{P(-1)} w_{(1)})$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ . We have a natural isomorphism  $\mathcal{R}$  from  $\boxtimes$  to  $\boxtimes \circ \sigma_{12}$ .

Let  $z_1$  and  $z_2$  be complex numbers satisfying  $|z_1| > |z_2| > |z_1 - z_2| > 0$  and let  $W_1$ ,  $W_2$  and  $W_3$  be objects of  $\mathcal{C}$ . Then the associativity isomorphism (corresponding to the indicated geometric data)

$$\alpha_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \rightarrow W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$$

and its inverse

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \rightarrow (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$$

have been constructed in Section 10 and are determined uniquely by (10.13) and (10.5), respectively.

Let  $z_1, z_2, z_3$  and  $z_4$  be any nonzero complex numbers (with no restrictions other than being nonzero; they may all be equal, for example) and let  $W_1$ ,  $W_2$  and  $W_3$  be objects of  $\mathcal{C}$ . In this generality, we define a natural associativity isomorphism

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_4), P(z_3)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \rightarrow (W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3$$

using the already-constructed associativity and parallel transport isomorphisms as follows: Let  $\zeta_1$  and  $\zeta_2$  be nonzero complex numbers satisfying  $|\zeta_1| > |\zeta_2| > |\zeta_1 - \zeta_2| > 0$ . Let  $\gamma_1$  and  $\gamma_2$ , be paths from  $z_1$  and  $z_2$  to  $\zeta_1$  and  $\zeta_2$ , respectively, in the complex plane with a cut along the positive real line, and let  $\gamma_3$  and  $\gamma_4$  be paths from  $\zeta_2$  and  $\zeta_1 - \zeta_2$  to  $z_3$  and  $z_4$ , respectively, also in the complex plane with a cut along the positive real line. Then we define

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_4), P(z_3)} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(\zeta_2)} 1_{W_3}) \circ \mathcal{A}_{P(\zeta_1), P(\zeta_2)}^{P(\zeta_1-\zeta_2), P(\zeta_2)} \circ (1_{W_1} \boxtimes_{P(\zeta_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1},$$

that is,  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_4), P(z_3)}$  is given by the commutative diagram

$$\begin{array}{ccc} W_1 \boxtimes_{P(\zeta_1)} (W_2 \boxtimes_{P(\zeta_2)} W_3) & \xrightarrow{\mathcal{A}_{P(\zeta_1), P(\zeta_2)}^{P(\zeta_1-\zeta_2), P(\zeta_2)}} & (W_1 \boxtimes_{P(\zeta_1-\zeta_2)} W_2) \boxtimes_{P(\zeta_2)} W_3 \\ \uparrow (1_{W_1} \boxtimes_{P(\zeta_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1} & & \downarrow \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(\zeta_2)} 1_{W_3}) \\ W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \xrightarrow{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_4), P(z_3)}} & (W_1 \boxtimes_{P(z_4)} W_2) \boxtimes_{P(z_3)} W_3 \end{array}$$

The inverse of  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_4), P(z_3)}$  is denoted  $\alpha_{P(z_1), P(z_2)}^{P(z_4), P(z_3)}$ . These isomorphisms certainly generalize the previously-constructed ones (when the appropriate inequalities hold).

In particular, when  $z_1 = z_2 = z_3 = z_4 = 1$ , we call the corresponding natural associativity isomorphism

$$\mathcal{A}_{P(1), P(1)}^{P(1), P(1)} : W_1 \boxtimes (W_2 \boxtimes W_3) \rightarrow (W_1 \boxtimes W_2) \boxtimes W_3$$

the *associativity isomorphism* (for the braided tensor category structure) and denote it simply by  $\mathcal{A}$ . Its inverse is denoted by  $\alpha$ . Because of the importance of this special case, we rewrite the definition of  $\mathcal{A}$  explicitly. Let  $r_1$  and  $r_2$  be real numbers satisfying  $r_1 > r_2 > r_1 - r_2 > 0$ . Let  $\gamma_1$  and  $\gamma_2$  be paths in  $(0, \infty)$  from 1 to  $r_1$  and to  $r_2$ , respectively, and let  $\gamma_3$  and  $\gamma_4$  be paths in  $(0, \infty)$  from  $r_2$  and from  $r_1 - r_2$  to 1, respectively. Then

$$\mathcal{A} = \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(r_2)} 1_{W_3}) \circ \mathcal{A}_{P(r_1), P(r_2)}^{P(r_1-r_2), P(r_2)} \circ (1_{W_1} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1},$$

that is,  $\mathcal{A}$  is given by the commutative diagram

$$\begin{array}{ccc} W_1 \boxtimes_{P(r_1)} (W_2 \boxtimes_{P(r_2)} W_3) & \xrightarrow{\mathcal{A}_{P(r_1), P(r_2)}^{P(r_1-r_2), P(r_2)}} & (W_1 \boxtimes_{P(r_1-r_2)} W_2) \boxtimes_{P(r_2)} W_3 \\ \uparrow (1_{W_1} \boxtimes_{P(r_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1} & & \downarrow \mathcal{T}_{\gamma_3} \circ (\mathcal{T}_{\gamma_4} \boxtimes_{P(r_2)} 1_{W_3}) \\ W_1 \boxtimes (W_2 \boxtimes W_3) & \xrightarrow{\mathcal{A}} & (W_1 \boxtimes W_2) \boxtimes W_3. \end{array}$$

**Remark 12.11** It is important to note that in this case, or more generally, whenever  $z_1 = z_2$  or  $z_3 = z_4$  (or both), the corresponding tensor products of *elements* fail to exist; for instance, the symbol  $w_{(1)} \boxtimes_{P(1)} (w_{(2)} \boxtimes_{P(1)} w_{(3)})$  has no meaning. Because of this, it has been necessary for us to develop our whole theory for general nonzero complex numbers (and this in turn has required the theory of the logarithmic operator product expansion and so on).

### 12.3 Actions of the associativity and commutativity isomorphisms on tensor products of elements

**Proposition 12.12** *For any  $z_1, z_2 \in \mathbb{C}^\times$  such that  $z_1 \neq z_2$  but  $|z_1| = |z_2| = |z_1 - z_2|$ , we have*

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}}(w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) = (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \quad (12.28)$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ , where

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \rightarrow \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}$$

is the natural extension of  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$ .

*Proof* We need only prove the case that  $w_{(1)}$  and  $w_{(2)}$  are homogeneous with respect to the generalized-weight grading. So we now assume that they are homogeneous.



We can always find  $\epsilon_1 \in \mathbb{C}$  such that

$$|z_1 + \epsilon_1| > |\epsilon_1|, \quad (12.29)$$

$$|z_1 + \epsilon_1| > |z_2| > |(z_1 + \epsilon_1) - z_2| > 0. \quad (12.30)$$

Let  $\mathcal{Y}_1 = \mathcal{Y}_{\boxtimes_{P(z_1)}, 0}$  and  $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes_{P(z_2)}, 0}$  be intertwining operators of types

$$\begin{pmatrix} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\ W_1 \quad W_2 \boxtimes_{P(z_2)} W_3 \end{pmatrix}$$

and

$$\begin{pmatrix} W_2 \boxtimes_{P(z_2)} W_3 \\ W_2 \quad W_3 \end{pmatrix},$$

respectively, corresponding to the intertwining maps  $\boxtimes_{P(z_1)}$  and  $\boxtimes_{P(z_2)}$ , respectively. Then the series

$$\langle w', \mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(\pi_n(w_{(2)}), z_2) w_{(3)} \rangle$$

is absolutely convergent for  $m, n \in \mathbb{R}$  and

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and the sums of these series define elements

$$\mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}.$$

By the definition of the parallel transport isomorphism, for any path  $\gamma$  from  $z_1 + \epsilon_1$  to  $z_1$  in the complex plane with a cut along the nonnegative real line, we have

$$\begin{aligned} & \overline{\mathcal{T}_\gamma}(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}) \boxtimes_{P(z_1 + \epsilon_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \\ &= \mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)}. \end{aligned} \quad (12.31)$$

By definition, we know that

$$\mathcal{T}_\gamma^{-1} = \mathcal{T}_{\gamma^{-1}},$$

so that (12.31) can be written as

$$\begin{aligned} & \overline{\mathcal{T}_{\gamma_1^{-1}}}(\mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)}) \\ &= \pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}) \boxtimes_{P(z_1 + \epsilon_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}). \end{aligned} \quad (12.32)$$

Since (12.30) holds, by (10.5), we have

$$\begin{aligned} & \overline{\mathcal{A}_{P(z_1 + \epsilon_1), P(z_2)}^{P((z_1 + \epsilon_1) - z_1), P(z_2)}}(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}) \boxtimes_{P(z_1 + \epsilon_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})) \\ &= (\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}) \boxtimes_{P((z_1 + \epsilon_1) - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}. \end{aligned} \quad (12.33)$$

Let  $\mathcal{Y}_3 = \mathcal{Y}_{\boxtimes_{P(z_2)}, 0}$  and  $\mathcal{Y}_4 = \mathcal{Y}_{\boxtimes_{P(z_1-z_2)}, 0}$  be intertwining operators of types

$$\begin{pmatrix} (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\ (W_1 \boxtimes_{P(z_1-z_2)} W_2) \quad W_3 \end{pmatrix}$$

and

$$\begin{pmatrix} W_1 \boxtimes_{P(z_1-z_2)} W_2 \\ W_1 \quad W_2 \end{pmatrix},$$

respectively, corresponding to the intertwining maps  $\boxtimes_{P(z_2)}$  and  $\boxtimes_{P(z_1-z_2)}$ , respectively. Then the series

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)} \rangle$$

is absolutely convergent for  $m, n \in \mathbb{R}$  and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)',$$

and the sums of these series define elements

$$\begin{aligned} & \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)} \\ & \in \overline{(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3}. \end{aligned}$$

We can always choose  $\gamma$  such that the path  $\gamma - z_2$  from  $(z_1 + \epsilon_1) - z_2$  to  $z_1 - z_2$  is also in the complex plane with a cut along the nonnegative real line. Then by the definition of the parallel transport isomorphism, we have

$$\begin{aligned} & \overline{\mathcal{T}_{\gamma-z_2}}((\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}) \boxtimes_{P((z_1+\epsilon_1)-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}) \\ & = \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)}. \end{aligned} \tag{12.34}$$

Combining (12.32)–(12.34) and using the definition of the associativity isomorphism  $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}$ , we obtain

$$\begin{aligned} & \overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1-z_2), P(z_2)}}(\mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)}) \\ & = \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)}. \end{aligned} \tag{12.35}$$

Since (12.30) holds, both the series

$$\sum_{m \in \mathbb{R}} \langle w', \mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \rangle$$

and

$$\sum_{m \in \mathbb{R}} \langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)} \rangle$$

are absolutely convergent for

$$w' \in (W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3))'$$

and

$$\tilde{w}' \in ((W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3)'.$$

We know that

$$\langle \tilde{w}', \mathcal{Y}_1(w_{(1)}, z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \rangle$$

and

$$\langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(w_{(1)}, (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)} \rangle$$

are the values of single-valued analytic functions

$$F(w', w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2)$$

and

$$G(\tilde{w}', w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2)$$

of  $\zeta_1$  and  $\zeta_2$  in a neighborhood of the point  $(\zeta_1, \zeta_2) = (z_1 + \epsilon_1, z_2)$  which contains the point  $(\zeta_1, \zeta_2) = (z_1, z_2)$ . Then by the definition of the tensor product elements  $w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)})$  and  $(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$ , we have

$$\langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle = F(w', w_{(1)}, w_{(2)}, w_{(3)}; z_1, z_2) \quad (12.36)$$

and

$$\langle \tilde{w}', (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle = G(\tilde{w}', w_{(1)}, w_{(2)}, w_{(3)}; z_1, z_2). \quad (12.37)$$

On the other hand, since  $F(w', w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2)$  and  $G(\tilde{w}', w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2)$  are analytic extensions of matrix elements of products and iterates of intertwining maps, properties of these products and iterates also hold for these functions if they still make sense. In particular, they satisfy the  $L(-1)$ -derivative property:

$$\frac{\partial}{\partial \zeta_1} F(w', w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2) = F(w', L(-1)w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2), \quad (12.38)$$

$$\frac{\partial}{\partial \zeta_1} G(\tilde{w}', w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2) = G(\tilde{w}', L(-1)w_{(1)}, w_{(2)}, w_{(3)}; \zeta_1, \zeta_2), \quad (12.39)$$

From the Taylor theorem (which applies since (12.29) holds) and (12.38)–(12.39), we have

$$F(w', w_{(1)}, w_{(2)}, w_{(3)}; z_1, z_2) = \sum_{m \in \mathbb{R}} F(w', \pi_m(e^{-\epsilon_1 L(-1)})w_{(1)}, w_{(2)}, w_{(3)}; z_1 + \epsilon_1, z_2), \quad (12.40)$$

$$G(\tilde{w}', w_{(1)}, w_{(2)}, w_{(3)}; z_1, z_2) = \sum_{m \in \mathbb{R}} G(\tilde{w}', \pi_m(e^{-\epsilon_1 L(-1)})w_{(1)}, w_{(2)}, w_{(3)}; z_1 + \epsilon_1, z_2). \quad (12.41)$$

Thus by the definitions of

$$\begin{aligned} & F(w', \pi_m(e^{-\epsilon_1 L(-1)})w_{(1)}, w_{(2)}, w_{(3)}; z_1 + \epsilon_1, z_2), \\ & G(\tilde{w}', \pi_m(e^{-\epsilon_1 L(-1)})w_{(1)}, w_{(2)}, w_{(3)}; z_1 + \epsilon_1, z_2), \end{aligned}$$

and by (12.40), (12.41), (12.36) and (12.37), we obtain

$$\begin{aligned} \sum_{m \in \mathbb{R}} \langle w', \mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \rangle \\ = \langle w', w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \rangle \end{aligned} \quad (12.42)$$

and

$$\begin{aligned} \sum_{m \in \mathbb{R}} \langle \tilde{w}', \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)} \rangle \\ = \langle \tilde{w}', (w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle. \end{aligned} \quad (12.43)$$

Since  $w'$  and  $\tilde{w}'$  are arbitrary, (12.42) and (12.43) gives

$$\begin{aligned} \sum_{m \in \mathbb{R}} \mathcal{Y}_1(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), z_1 + \epsilon_1) \mathcal{Y}_2(w_{(2)}, z_2) w_{(3)} \\ = w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} w_{(3)}) \end{aligned} \quad (12.44)$$

and

$$\begin{aligned} \sum_{m \in \mathbb{R}} \mathcal{Y}_3(\mathcal{Y}_4(\pi_m(e^{-\epsilon_1 L(-1)} w_{(1)}), (z_1 + \epsilon_1) - z_2) w_{(2)}, z_2) w_{(3)} \\ = (w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}. \end{aligned} \quad (12.45)$$

Taking the sum  $\sum_{m \in \mathbb{R}}$  on both sides of (12.35) and then using (12.44) and (12.45), we obtain (12.28).  $\square$

We also have:

**Proposition 12.13** *Let  $z_1, z_2$  be nonzero complex numbers such that  $z_1 \neq z_2$  but  $|z_1| = |z_2| = |z_1 - z_2|$ . Let  $\gamma$  be a path from  $z_2$  to  $z_1$  in the complex plane with a cut along the nonnegative real line. Then we have*

$$\overline{\mathcal{T}_\gamma \circ (\mathcal{R}_{P(z_1 - z_2)} \boxtimes_{P(z_2)} 1_{W_3})} ((w_{(1)} \boxtimes_{P(z_1 - z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}) = (w_{(2)} \boxtimes_{P(z_2 - z_1)} w_{(1)}) \boxtimes_{P(z_1)} w_{(3)} \quad (12.46)$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .

*Proof* We can find  $\epsilon$  such that  $|z_2| > |\epsilon|$ ,

$$|z_2 + \epsilon| > |z_1 - z_2| > 0.$$

Let  $\mathcal{Y}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)}, 0}$ ,  $\tilde{\mathcal{Y}}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)}, 0}$  and  $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes_{P(z_1 - z_2)}, 0}$  be intertwining operators of types

$$\begin{pmatrix} (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\ (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \quad W_3 \end{pmatrix},$$

$$\begin{pmatrix} (W_2 \boxtimes_{P(z_2-z_1)} W_1) \boxtimes_{P(z_2)} W_3 \\ (W_2 \boxtimes_{P(z_2-z_1)} W_1) \quad W_3 \end{pmatrix},$$

and

$$\begin{pmatrix} W_1 \boxtimes_{P(z_1-z_2)} W_2 \\ W_1 \quad W_2 \end{pmatrix},$$

respectively, corresponding to the intertwining maps  $\boxtimes_{P(z_2)}$ ,  $\boxtimes_{P(z_2)}$  and  $\boxtimes_{P(z_1)}$ , respectively. Note that since  $|z_2 + \epsilon| > |z_1 - z_2| > 0$ ,

$$\begin{aligned} & \mathcal{Y}_1(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}, z_2 + \epsilon) w_{(3)} \\ &= \sum_{m,n \in \mathbb{R}} \sum_{k=1}^N (\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}))_{m;k}^{\mathcal{Y}_1} w_{(3)} e^{(-m-1) \log(z_2+\epsilon)} (\log(z_2 + \epsilon))^k. \end{aligned} \quad (12.47)$$

For  $n \in \mathbb{R}$ ,

$$\begin{aligned} & \sum_m \sum_{k=1}^N (\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}))_{m;k}^{\mathcal{Y}_1} w_{(3)} e^{(-m-1) \log z_2} (\log z_2)^k \\ &= (\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)})) \boxtimes_{P(z_2)} w_{(3)}. \end{aligned} \quad (12.48)$$

But by the definition of  $\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3}$ ,

$$\begin{aligned} & \overline{(\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})} ((\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)})) \boxtimes_{P(z_2)} w_{(3)}) \\ &= \mathcal{R}_{P(z_1-z_2)} (\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)})) \boxtimes_{P(z_2)} w_{(3)} \\ &= \pi_n(\mathcal{R}_{P(z_1-z_2)} (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)})) \boxtimes_{P(z_2)} w_{(3)} \end{aligned} \quad (12.49)$$

for  $n \in \mathbb{R}$ . From (12.48), (12.49) and the definitions of  $\mathcal{Y}_1$  and  $\tilde{\mathcal{Y}}_1$ , we obtain

$$\begin{aligned} & \overline{(\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})} ((\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}))_{m;k}^{\mathcal{Y}_1} w_{(3)}) \\ &= (\pi_n(\mathcal{R}_{P(z_1-z_2)} (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)})))_{m;k}^{\tilde{\mathcal{Y}}_1} w_{(3)} \\ &= (\pi_n(e^{(z_1-z_2)L(-1)} (w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})))_{m;k}^{\tilde{\mathcal{Y}}_1} w_{(3)} \end{aligned} \quad (12.50)$$

for  $m \in \mathbb{R}$  and  $k = 1, \dots, N$ . Using (12.47) and (12.50), we obtain

$$\begin{aligned} & \overline{(\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})} (\mathcal{Y}_1(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}, z_2 + \epsilon) w_{(3)}) \\ &= \overline{(\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})} \left( \sum_{m,n \in \mathbb{R}} \sum_{k=1}^N (\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}))_{m;k}^{\mathcal{Y}_1} w_{(3)} e^{(-m-1) \log(z_2+\epsilon)} (\log(z_2 + \epsilon))^k \right) \\ &= \sum_{m,n \in \mathbb{R}} \sum_{k=1}^N \overline{(\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})} ((\pi_n(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}))_{m;k}^{\mathcal{Y}_1} w_{(3)}) \cdot \\ & \quad \cdot e^{(-m-1) \log(z_2+\epsilon)} (\log(z_2 + \epsilon))^k \end{aligned}$$

$$\begin{aligned}
&= \sum_{m,n \in \mathbb{R}} \sum_{k=1}^N (\pi_n(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})))_{m;k}^{\tilde{\mathcal{Y}}_1} w_{(3)} \cdot \\
&\quad \cdot e^{(-m-1)\log(z_2+\epsilon)} (\log(z_2+\epsilon))^k \\
&= \tilde{\mathcal{Y}}_1(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_2+\epsilon) w_{(3)}. \tag{12.51}
\end{aligned}$$

Let  $\mathcal{Y}_3 = \mathcal{Y}_{\boxtimes_{P(z_1)},0}$  be the intertwining operator of type

$$\begin{pmatrix} (W_2 \boxtimes_{P(z_2-z_1)} W_1) \boxtimes_{P(z_1)} W_3 \\ (W_2 \boxtimes_{P(z_2-z_1)} W_1) W_3 \end{pmatrix}.$$

Then by the definition of the parallel transport isomorphism, for  $m \in \mathbb{R}$  we have

$$\begin{aligned}
&\overline{\mathcal{T}}_\gamma(\tilde{\mathcal{Y}}_1(\pi_m(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})), z_2+\epsilon) w_{(3)}) \\
&= \overline{\mathcal{T}}_\gamma(\tilde{\mathcal{Y}}_1(e^{\epsilon L(-1)} \pi_m(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})), z_2) w_{(3)}) \\
&= \overline{\mathcal{T}}_\gamma((e^{\epsilon L(-1)} \pi_m(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}))) \boxtimes_{P(z_2)} w_{(3)}) \\
&= \mathcal{Y}_3(e^{\epsilon L(-1)} \pi_m(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})), z_2) w_{(3)} \\
&= \mathcal{Y}_3(\pi_m(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})), z_2+\epsilon) w_{(3)}. \tag{12.52}
\end{aligned}$$

Since  $|z_2+\epsilon| > |z_1-z_2| > 0$ , the sums of both sides of (12.52) for  $m \in \mathbb{R}$  are absolutely convergent (in the sense that the series obtained by paring it with elements of  $((W_2 \boxtimes_{P(z_2-z_1)} W_1) \boxtimes_{P(z_1)} W_3)'$  are absolutely convergent) and we have

$$\begin{aligned}
&\sum_{m \in \mathbb{R}} \overline{\mathcal{T}}_\gamma(\tilde{\mathcal{Y}}_1(\pi_m(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})), z_2+\epsilon) w_{(3)}) \\
&= \overline{\mathcal{T}}_\gamma(\tilde{\mathcal{Y}}_1(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_2+\epsilon) w_{(3)}) \tag{12.53}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m \in \mathbb{R}} \mathcal{Y}_3(\pi_m(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)})), z_2+\epsilon) w_{(3)} \\
&= \mathcal{Y}_3((w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_2+\epsilon+(z_1-z_2)) w_{(3)} \\
&= \mathcal{Y}_3((w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_1+\epsilon) w_{(3)}. \tag{12.54}
\end{aligned}$$

From (12.52)–(12.54), we obtain

$$\begin{aligned}
&\overline{\mathcal{T}}_\gamma(\tilde{\mathcal{Y}}_1(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_2+\epsilon) w_{(3)}) \\
&= \mathcal{Y}_3((w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_1+\epsilon) w_{(3)}. \tag{12.55}
\end{aligned}$$

From (12.51) and (12.55), we obtain

$$\begin{aligned}
&\overline{\mathcal{T}}_\gamma \circ (\overline{\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3}})(\mathcal{Y}_1(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}, z_2+\epsilon) w_{(3)}) \\
&= \overline{\mathcal{T}}_\gamma(\tilde{\mathcal{Y}}_1(e^{(z_1-z_2)L(-1)}(w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_2+\epsilon) w_{(3)}) \\
&= \mathcal{Y}_3((w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_1+\epsilon) w_{(3)}.
\end{aligned}$$

Then for any

$$w' \in ((W_1 \boxtimes_{P(z_2-z_1)} W_2) \boxtimes_{P(z_1)} W_3)',$$

we have

$$\begin{aligned} & \langle w', \overline{\mathcal{T}_\gamma \circ (\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})} (\mathcal{Y}_1(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}, z_2 + \epsilon) w_{(3)}) \rangle \\ &= \langle w', \mathcal{Y}_3((w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_1 + \epsilon) w_{(3)} \rangle, \end{aligned}$$

or equivalently,

$$\begin{aligned} & \langle ((\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})' \circ \mathcal{T}_\gamma')(w'), \mathcal{Y}_1(w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}, z_2 + \epsilon) w_{(3)} \rangle \\ &= \langle w', \mathcal{Y}_3((w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}), z_1 + \epsilon) w_{(3)} \rangle, \end{aligned} \quad (12.56)$$

The left- and right-hand sides of (12.56) are values at  $(\zeta_1, \zeta_2) = (z_1 + \epsilon, z_2 + \epsilon)$  of some single-valued analytic functions of  $\zeta_1$  and  $\zeta_2$  defined in the region

$$\{(\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid \zeta_1 \neq 0, \zeta_2 \neq 0, \zeta_1 \neq \zeta_2, 0 \leq \arg \zeta_1, \arg \zeta_2, \arg(\zeta_1 - \zeta_2) < 2\pi\}.$$

Also, by the definition of tensor product of three elements above, the values of these analytic functions at  $(\zeta_1, \zeta_2) = (z_1, z_2)$  are equal to

$$\langle ((\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})' \circ \mathcal{T}_\gamma')(w'), (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle$$

and

$$\langle w', (w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}) \boxtimes_{P(z_1)} w_{(3)} \rangle,$$

respectively. Thus we can take the limit  $\epsilon \rightarrow 0$  on both sides of (12.56) and obtain

$$\begin{aligned} & \langle ((\mathcal{R}_{P(z_1-z_2)} \boxtimes_{P(z_2)} 1_{W_3})' \circ \mathcal{T}_\gamma')(w'), (w_{(1)} \boxtimes_{P(z_1-z_2)} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)} \rangle \\ &= \langle w', (w_{(2)} \boxtimes_{P(z_2-z_1)} w_{(1)}) \boxtimes_{P(z_1)} w_{(3)} \rangle. \end{aligned} \quad (12.57)$$

Since  $w'$  is arbitrary, (12.57) is equivalent to (12.46).  $\square$

We also prove:

**Proposition 12.14** *Let  $z_1, z_2$  be nonzero complex numbers such that  $z_1 \neq z_2$  but  $|z_1| = |z_2| = |z_1 - z_2|$ . Let  $\gamma$  be a path from  $z_2$  to  $z_2 - z_1$  in the complex plane with a cut along the nonnegative real line. Then we have*

$$\begin{aligned} & \overline{\mathcal{T}_\gamma \circ (1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})} (w_{(2)} \boxtimes_{P(z_2)} (w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \\ &= e^{z_1 L(-1)} (w_{(2)} \boxtimes_{P(z_2-z_1)} (w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) \end{aligned} \quad (12.58)$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ .

*Proof* We can find  $\epsilon$  such that  $|z_2| > |\epsilon|$ ,

$$|z_2 + \epsilon|, |z_2 - z_1 + \epsilon| > |z_1| > 0.$$

Let  $\mathcal{Y}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)}, 0}$ ,  $\tilde{\mathcal{Y}}_1 = \mathcal{Y}_{\boxtimes_{P(z_2)}, 0}$  and  $\mathcal{Y}_2 = \mathcal{Y}_{\boxtimes_{P(z_1)}, 0}$  be intertwining operators of types

$$\begin{pmatrix} W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) \\ W_2 \quad (W_1 \boxtimes_{P(z_1)} W_3) \end{pmatrix},$$

$$\begin{pmatrix} W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) \\ W_2 \quad (W_3 \boxtimes_{P(-z_1)} W_1) \end{pmatrix}$$

and

$$\begin{pmatrix} W_1 \boxtimes_{P(z_1)} W_3 \\ W_1 \quad W_3 \end{pmatrix},$$

respectively, corresponding to the intertwining maps  $\boxtimes_{P(z_2)}$ ,  $\boxtimes_{P(z_2)}$  and  $\boxtimes_{P(z_1)}$ , respectively. Since  $|z_2 + \epsilon| > |z_1| > 0$ ,

$$\begin{aligned} & \mathcal{Y}_1(w_{(2)}, z_2 + \epsilon)(w_{(1)} \boxtimes_{P(z_1)} w_{(3)}) \\ &= \sum_{m, n \in \mathbb{R}} \sum_{k=1}^N (w_{(2)})_{m; k}^{\mathcal{Y}_1} \pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)}) e^{(-m-1) \log(z_2 + \epsilon)} (\log(z_2 + \epsilon))^k. \end{aligned} \quad (12.59)$$

For  $n \in \mathbb{R}$ ,

$$\begin{aligned} & \sum_{m \in \mathbb{R}} \sum_{k=1}^N (w_{(2)})_{m; k}^{\mathcal{Y}_1} \pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)}) e^{(-m-1) \log z_2} (\log z_2)^k \\ &= w_{(2)} \boxtimes_{P(z_2)} \pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)}). \end{aligned} \quad (12.60)$$

But by the definition of  $1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)}$ ,

$$\begin{aligned} & \overline{(1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})} (w_{(2)} \boxtimes_{P(z_2)} \pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \\ &= w_{(2)} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)} (\pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \\ &= w_{(2)} \boxtimes_{P(z_2)} \pi_n(\mathcal{R}_{P(z_1)}(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \end{aligned} \quad (12.61)$$

for  $n \in \mathbb{R}$ . From (12.60), (12.61) and the definitions of  $\mathcal{Y}_1$  and  $\tilde{\mathcal{Y}}_1$ , we obtain

$$\begin{aligned} & \overline{(1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})} ((w_{(2)})_{m; k}^{\mathcal{Y}_1} \pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \\ &= (w_{(2)})_{m; k}^{\tilde{\mathcal{Y}}_1} \pi_n(\mathcal{R}_{P(z_1)}(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \\ &= (w_{(2)})_{m; k}^{\tilde{\mathcal{Y}}_1} \pi_n(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) \end{aligned} \quad (12.62)$$



for  $m \in \mathbb{R}$  and  $k = 1, \dots, N$ . Using (12.59) and (12.62), we obtain

$$\begin{aligned}
& \overline{(1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})} (\mathcal{Y}_1(w_{(2)}, z_2 + \epsilon)(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \\
&= \overline{(1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})} \\
& \quad \left( \sum_{m,n \in \mathbb{R}} \sum_{k=1}^N (w_{(2)})_{m;k}^{\mathcal{Y}_1} \pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)}) e^{(-m-1) \log(z_2 + \epsilon)} (\log(z_2 + \epsilon))^k \right) \\
&= \sum_{m,n \in \mathbb{R}} \sum_{k=1}^N \overline{(1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})} ((w_{(2)})_{m;k}^{\mathcal{Y}_1} \pi_n(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \cdot \\
& \quad \cdot e^{(-m-1) \log(z_2 + \epsilon)} (\log(z_2 + \epsilon))^k \\
&= \sum_{m,n \in \mathbb{R}} \sum_{k=1}^N (w_{(2)})_{m;k}^{\tilde{\mathcal{Y}}_1} \pi_n(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) e^{(-m-1) \log(z_2 + \epsilon)} (\log(z_2 + \epsilon))^k \\
&= \tilde{\mathcal{Y}}_1(w_{(2)}, z_2 + \epsilon)(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})). \tag{12.63}
\end{aligned}$$

Let  $\mathcal{Y}_3 = \mathcal{Y}_{\boxtimes_{P(z_1)}, 0}$  be the intertwining operator of type

$$\begin{pmatrix} W_2 \boxtimes_{P(z_2-z_1)} (W_1 \boxtimes_{P(-z_1)} W_3) \\ W_2 \quad W_1 \boxtimes_{(-z_1)} W_3 \end{pmatrix}.$$

Then by the definition of the parallel transport isomorphism, for  $m \in \mathbb{R}$  we have

$$\begin{aligned}
& \overline{\mathcal{T}_\gamma}(\tilde{\mathcal{Y}}_1(w_{(2)}, z_2 + \epsilon) \pi_m(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}))) \\
&= \overline{\mathcal{T}_\gamma}(e^{\epsilon L(-1)} \tilde{\mathcal{Y}}_1(w_{(2)}, z_2) e^{\epsilon L(-1)} \pi_m(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}))) \\
&= \overline{\mathcal{T}_\gamma}(e^{\epsilon L(-1)} w_{(2)} \boxtimes_{P(z_2)} e^{\epsilon L(-1)} \pi_m(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}))) \\
&= e^{\epsilon L(-1)} \mathcal{Y}_3(w_{(2)}, z_2) e^{\epsilon L(-1)} \pi_m(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) \\
&= \mathcal{Y}_3(w_{(2)}, z_2 + \epsilon) \pi_m(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})). \tag{12.64}
\end{aligned}$$

Since  $|z_2 + \epsilon| > |z_1| > 0$ , the sums of both sides of (12.64) over  $m \in \mathbb{R}$  are absolutely convergent. Note also that  $|z_2 - z_1 + \epsilon| > |z_1| > 0$ . Thus we have

$$\begin{aligned}
& \sum_{m \in \mathbb{R}} \overline{\mathcal{T}_\gamma}(\tilde{\mathcal{Y}}_1(w_{(2)}, z_2 + \epsilon) \pi_m(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}))) \\
&= \overline{\mathcal{T}_\gamma}(\tilde{\mathcal{Y}}_1(w_{(2)}, z_2 + \epsilon) e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) \\
&= \overline{\mathcal{T}_\gamma}(e^{z_1 L(-1)} \tilde{\mathcal{Y}}_1(w_{(2)}, z_2 - z_1 + \epsilon)(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) \tag{12.65}
\end{aligned}$$

and

$$\begin{aligned}
& \sum_{m \in \mathbb{R}} \mathcal{Y}_3(w_{(2)}, z_2 + \epsilon) \pi_m(e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) \\
&= \mathcal{Y}_3(w_{(2)}, z_2 + \epsilon) e^{z_1 L(-1)}(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}) \\
&= e^{z_1 L(-1)} \mathcal{Y}_3(w_{(2)}, z_2 - z_1 + \epsilon)(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}). \tag{12.66}
\end{aligned}$$

From (12.64)–(12.66), we obtain

$$\begin{aligned} & \overline{\mathcal{T}_\gamma}(e^{z_1 L(-1)} \mathcal{Y}_1(w_{(2)}, z_2 - z_1 + \epsilon)(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})) \\ &= e^{z_1 L(-1)} \mathcal{Y}_3(w_{(2)}, z_2 - z_1 + \epsilon)(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}). \end{aligned} \quad (12.67)$$

From (12.63) and (12.67), we obtain

$$\begin{aligned} & \overline{\mathcal{T}_\gamma \circ (1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})}(\mathcal{Y}_1(w_{(2)}, z_2 + \epsilon)(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \\ &= e^{z_1 L(-1)} \mathcal{Y}_3(w_{(2)}, z_2 - z_1 + \epsilon)(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}). \end{aligned} \quad (12.68)$$

For any

$$w' \in (W_2 \boxtimes_{P(z_2-z_1)} (W_1 \boxtimes_{P(-z_1)} W_3))',$$

we have

$$\begin{aligned} & \langle w', \overline{\mathcal{T}_\gamma \circ (1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)})}(\mathcal{Y}_1(w_{(2)}, z_2 + \epsilon)(w_{(1)} \boxtimes_{P(z_1)} w_{(3)})) \rangle \\ &= \langle w', e^{z_1 L(-1)} \mathcal{Y}_3(w_{(2)}, z_2 - z_1 + \epsilon)(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}) \rangle, \end{aligned}$$

or equivalently,

$$\begin{aligned} & \langle ((1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)}) \circ \mathcal{T}_\gamma)(w)', \mathcal{Y}_1(w_{(2)}, z_2 + \epsilon)(w_{(1)} \boxtimes_{P(z_1)} w_{(3)}) \rangle \\ &= \langle w', e^{z_1 L(-1)} \mathcal{Y}_3(w_{(2)}, z_2 - z_1 + \epsilon)(w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}) \rangle, \end{aligned} \quad (12.69)$$

The left- and right-hand sides of (12.69) are values at  $(\zeta_1, \zeta_2) = (z_1, z_2 + \epsilon)$  of single-valued analytic functions of  $\zeta_1$  and  $\zeta_2$  defined in the region

$$\{(\zeta_1, \zeta_2) \in \mathbb{C}^2 \mid \zeta_1 \neq 0, \zeta_2 \neq 0, \zeta_1 \neq \zeta_2, 0 \leq \arg \zeta_1, \arg \zeta_2, \arg(\zeta_1 - \zeta_2) < 2\pi\}.$$

Also, by the definition of tensor product of three elements above, the values of these analytic functions at  $(\zeta_1, \zeta_2) = (z_1, z_2)$  are equal to

$$\langle ((1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)}) \circ \mathcal{T}_\gamma)(w)', w_{(2)} \boxtimes_{P(z_2)} (w_{(1)} \boxtimes_{P(z_1)} w_{(3)}) \rangle$$

and

$$\langle w', e^{z_1 L(-1)} w_{(2)} \boxtimes_{P(z_2-z_1)} (w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}) \rangle,$$

respectively. Thus we can take the limit  $\epsilon \rightarrow 0$  on both sides of (12.69) and obtain

$$\begin{aligned} & \langle ((1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)}) \circ \mathcal{T}_\gamma)(w)', w_{(2)} \boxtimes_{P(z_2)} (w_{(1)} \boxtimes_{P(z_1)} w_{(3)}) \rangle \\ &= \langle w', e^{z_1 L(-1)} w_{(2)} \boxtimes_{P(z_2-z_1)} (w_{(3)} \boxtimes_{P(-z_1)} w_{(1)}) \rangle. \end{aligned} \quad (12.70)$$

Since  $w'$  is arbitrary, (12.70) is equivalent to (12.58).  $\square$

## 12.4 The coherence properties

We shall use the following terminology concerning tensor categories; cf. [M], [T] and [BK]:

A *preadditive category* (or *Ab-category*) is a category in which each hom-set is an additive abelian group such that the composition of morphisms is bilinear. An *additive category* is a preadditive category which has a zero object and a biproduct for each pair of objects. An *abelian category* is an additive category such that every morphism has a kernel and a cokernel, every monic morphism is a kernel and every epic morphism is a cokernel.

A *monoidal category* is a category  $\mathcal{C}$  equipped with a *monoidal* or *tensor product bifunctor*  $\boxtimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , a *unit object*  $V$ , a *natural associativity isomorphism*  $\mathcal{A} : \boxtimes \circ (1_{\mathcal{C}} \times \boxtimes) \rightarrow \boxtimes \circ (\boxtimes \times 1_{\mathcal{C}})$ , a *natural left identity isomorphism*  $l : V \boxtimes \cdot \rightarrow 1_{\mathcal{C}}$ , and a *natural right identity isomorphism*  $r : \cdot \boxtimes V \rightarrow 1_{\mathcal{C}}$ , such that the *pentagon diagram*

$$\begin{array}{ccc}
 & W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \\
 \swarrow & & \searrow \\
 (W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) & & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \\
 \downarrow & & \downarrow \\
 ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4 & \xleftarrow{\quad} & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4
 \end{array} \tag{12.71}$$

and the *triangle diagram*

$$\begin{array}{ccc}
 (W_1 \boxtimes V) \boxtimes W_2 & \longrightarrow & W_1 \boxtimes (V \boxtimes W_2) \\
 \downarrow & & \downarrow \\
 W_1 \boxtimes W_2 & \xrightarrow{\quad = \quad} & W_1 \boxtimes W_2
 \end{array} \tag{12.72}$$

commute and such that the morphisms  $l_V : V \boxtimes V \rightarrow V$  and  $r_V : V \boxtimes V \rightarrow V$  are equal.

A *braided monoidal category* is a monoidal category with a *natural braiding isomorphism*

$$\mathcal{R} : \boxtimes \rightarrow \boxtimes \circ \sigma_{12},$$

where  $\sigma_{12}$  is the permutation functor on  $\mathcal{C} \times \mathcal{C}$ , such that the two hexagon diagrams (with  $\mathcal{R}^{\pm 1}$ )

$$\begin{array}{ccc}
& (W_1 \boxtimes W_2) \boxtimes W_3 & \\
\swarrow \mathcal{R}^{\pm 1} \boxtimes 1_{W_3} & & \searrow \mathcal{A}^{-1} \\
(W_2 \boxtimes W_1) \boxtimes W_3 & & W_1 \boxtimes (W_2 \boxtimes W_3) \\
\downarrow \mathcal{A}^{-1} & & \downarrow \mathcal{R}^{\pm 1} \\
W_2 \boxtimes (W_1 \boxtimes W_3) & & (W_2 \boxtimes W_3) \boxtimes W_1 \\
\searrow 1_{W_2} \boxtimes \mathcal{R}^{\pm 1} & & \swarrow \mathcal{A}^{-1} \\
& W_2 \boxtimes (W_3 \boxtimes W_1) &
\end{array} \tag{12.73}$$

commute.

An *additive braided monoidal category* is an additive category with a compatible braided monoidal category structure. A *tensor category* is an abelian category with a compatible monoidal category structure. A *braided tensor category* is an abelian category with a compatible braided monoidal category structure, that is, a tensor category with a compatible braiding structure.

**Theorem 12.15** *Let  $V$  be a Möbius or conformal vertex algebra and  $\mathcal{C}$  a full subcategory of  $\mathcal{M}_{sg}$  or  $\mathcal{GM}_{sg}$  (recall Notation 2.36) satisfying Assumptions 10.1, 12.1 and 12.2. Then the category  $\mathcal{C}$ , equipped with the tensor product bifunctor  $\boxtimes$ , the unit object  $V$ , the braiding isomorphisms  $\mathcal{R}$ , the associativity isomorphisms  $\mathcal{A}$ , and the left and right unit isomorphisms  $l$  and  $r$ , is an additive braided monoidal category.*

*Proof* We need only prove the coherence properties. We prove the commutativity of the pentagon diagram first. Let  $W_1, W_2, W_3$  and  $W_4$  be objects of  $\mathcal{C}$  and let  $z_1, z_2, z_3 \in \mathbb{R}$  such that

$$\begin{aligned}
|z_1| &> |z_2| > |z_3| > |z_{13}| > |z_{23}| > |z_{12}| > 0, \\
|z_1| &> |z_3| + |z_{23}| > 0, \\
|z_2| &> |z_{12}| + |z_3| > 0, \\
|z_3| &> |z_{23}| + |z_{12}| > 0,
\end{aligned} \tag{12.74}$$

where  $z_{12} = z_1 - z_2$ ,  $z_{13} = z_1 - z_3$  and  $z_{23} = z_2 - z_3$ . For example, we can take  $z_1 = 7$ ,  $z_2 = 6$  and  $z_3 = 4$ . We first prove the commutativity of the following diagram:

$$\begin{array}{ccc}
& W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) & \\
& \swarrow \qquad \searrow & \\
(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) & & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4) \\
\downarrow & & \downarrow \\
((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 & \longleftarrow & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4
\end{array} \tag{12.75}$$

For  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$  and  $w_{(4)} \in W_4$ , we consider

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)})) \in \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4))}.$$

By the characterizations of the associativity isomorphisms, we see that the compositions of the natural extensions of the module maps in the two routes in (12.75) applied to this element both give

$$((w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_{23})} w_{(3)}) \boxtimes_{P(z_3)} w_{(4)} \in \overline{((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4}.$$

Since the homogeneous components of

$$w_{(1)} \boxtimes_{P(z_1)} (w_{(2)} \boxtimes_{P(z_2)} (w_{(3)} \boxtimes_{P(z_3)} w_{(4)}))$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$ ,  $w_{(3)} \in W_3$  and  $w_{(4)} \in W_4$  span

$$W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)),$$

the diagram (12.75) above commutes.

On the other hand, by the definition of  $\mathcal{A}$ , the diagrams

$$\begin{array}{ccc}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) & \longrightarrow & (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) \\
\downarrow & & \downarrow \\
W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4)
\end{array} \tag{12.76}$$

$$\begin{array}{ccc}
(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) & \longrightarrow & ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 \\
\downarrow & & \downarrow \\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) & \longrightarrow & ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4
\end{array} \tag{12.77}$$

$$\begin{array}{ccc}
W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) & \longrightarrow & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4) \\
\downarrow & & \downarrow \\
W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \longrightarrow & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4)
\end{array} \tag{12.78}$$

$$\begin{array}{ccc}
W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4) & \longrightarrow & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 \\
\downarrow & & \downarrow \\
W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) & \longrightarrow & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4
\end{array} \tag{12.79}$$

$$\begin{array}{ccc}
(W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 & \longrightarrow & ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 \\
\downarrow & & \downarrow \\
(W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4 & \longrightarrow & ((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4
\end{array} \tag{12.80}$$

all commute. Combining all the diagrams (12.75)–(12.80) above, we see that the pentagon diagram

$$\begin{array}{ccc}
& W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \\
& \swarrow \quad \searrow & \\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) & & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \\
\downarrow & & \downarrow \\
((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4 & \longleftarrow & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4
\end{array}$$

also commutes.

Next we prove the commutativity of the hexagon diagrams. We prove only the commutativity of the hexagon diagram involving  $\mathcal{R}$ ; the proof of the commutativity of the other hexagon diagram is the same. Let  $W_1, W_2$  and  $W_3$  be objects of  $\mathcal{C}$  and let  $z_1, z_2 \in \mathbb{C}^\times$  satisfying  $|z_1| = |z_2| = |z_1 - z_2|$  and let  $z_{12} = z_1 - z_2$ . We first prove the commutativity of the following diagram:

$$\begin{array}{ccc}
& (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \\
\swarrow \mathcal{R}_{P(z_{12})} \boxtimes_{P(z_2)} 1_{W_3} & & \searrow \left( \mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)} \right)^{-1} \\
(W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & & W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \\
\downarrow \mathcal{T}_{\gamma_1} & & \downarrow \mathcal{R}_{P(z_1)} \\
(W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & & (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 \\
\downarrow \left( \mathcal{A}_{P(z_2), P(z_1)}^{P(-z_{12}), P(z_1)} \right)^{-1} & & \swarrow \left( \mathcal{A}_{P(-z_{12}), P(-z_1)}^{P(z_2), P(-z_1)} \right)^{-1} \\
W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & & W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) \\
\searrow 1_{W_2} \boxtimes_{P(z_2)} \mathcal{R}_{P(z_1)} & & \swarrow \mathcal{T}_{\gamma_2} \\
& W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) &
\end{array} \tag{12.81}$$

where  $\gamma_1$  and  $\gamma_2$  are paths from  $z_2$  to  $z_1$  and from  $-z_{12}$  to  $z_2$ , respectively, in  $\mathbb{C}$  with a cut along the nonnegative real line.

Let  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$ . By the results proved in the preceding section, we see that the images of the element

$$(w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$$

under the natural extension to

$$\overline{(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3}$$

of the compositions of the maps in both the left and right routes in (12.81) from

$$(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3$$

to

$$W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1)$$

are

$$w_{(2)} \boxtimes_{P(z_2)} (e^{z_1 L(-1)} (w_{(3)} \boxtimes_{P(-z_1)} w_{(1)})).$$

Since the homogeneous components of

$$(w_{(1)} \boxtimes_{P(z_{12})} w_{(2)}) \boxtimes_{P(z_2)} w_{(3)}$$

for  $w_{(1)} \in W_1$ ,  $w_{(2)} \in W_2$  and  $w_{(3)} \in W_3$  span

$$(W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3,$$

the diagram (12.81) commutes.

Now we consider the following diagrams:

$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes W_3 \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \end{array} \quad (12.82)$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \end{array} \quad (12.83)$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(-z_{12})} W_1) \boxtimes_{P(z_1)} W_3 & \longrightarrow & (W_2 \boxtimes W_1) \boxtimes W_3 \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & \longrightarrow & W_2 \boxtimes (W_1 \boxtimes W_3) \end{array} \quad (12.84)$$

$$\begin{array}{ccc} W_2 \boxtimes_{P(z_2)} (W_1 \boxtimes_{P(z_1)} W_3) & \longrightarrow & W_2 \boxtimes (W_1 \boxtimes W_3) \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \end{array} \quad (12.85)$$

$$\begin{array}{ccc} (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes W_3 \\ \downarrow & & \downarrow \\ W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \longrightarrow & W_1 \boxtimes (W_1 \boxtimes W_3) \end{array} \quad (12.86)$$

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \longrightarrow & W_1 \boxtimes (W_1 \boxtimes W_3) \\ \downarrow & & \downarrow \\ (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 & \longrightarrow & (W_2 \boxtimes W_3) \boxtimes W_1 \end{array} \quad (12.87)$$

$$\begin{array}{ccc} (W_2 \boxtimes_{P(z_2)} W_3) \boxtimes_{P(-z_1)} W_1 & \longrightarrow & (W_2 \boxtimes W_3) \boxtimes W_1 \\ \downarrow & & \downarrow \\ W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \end{array} \quad (12.88)$$



$$\begin{array}{ccc}
W_2 \boxtimes_{P(-z_{12})} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1) \\
\downarrow & & \downarrow \\
W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(-z_1)} W_1) & \longrightarrow & W_2 \boxtimes (W_3 \boxtimes W_1)
\end{array} \tag{12.89}$$

The commutativity of the diagrams (12.82), (12.85) and (12.87) follows from the definition of the commutativity isomorphism for the braided tensor category structure and the naturality of the parallel transport isomorphisms. The commutativity of (12.84), (12.86) and (12.88) follows from the definition of the associativity isomorphism for the braided tensor product structure. The commutativity of (12.83) and (12.89) follows from the facts that compositions of parallel transport isomorphisms are equal to the parallel transport isomorphisms associated to the products of the paths and that parallel transport isomorphisms associated to homotopically equivalent paths are equal. The commutativity of the hexagon diagram involving  $\mathcal{R}$  follows from (12.81)–(12.89).

Finally, we prove the commutativity of the triangle diagram for the unit isomorphisms. Let  $z_1$  and  $z_2$  be complex numbers such that  $|z_1| > |z_2| > |z_1 - z_2| > 0$  and let  $z_{12} = z_1 - z_2$ . Also let  $\gamma$  be a path from  $z_2$  to  $z_1$  in  $\mathbb{C}$  with a cut along the nonnegative real line. We first prove the commutativity of the following diagram:

$$\begin{array}{ccc}
(W_1 \boxtimes_{P(z_{12})} V) \boxtimes_{P(z_2)} W_2 & \xrightarrow{(\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)})^{-1}} & W_1 \boxtimes_{P(z_1)} (V \boxtimes_{P(z_2)} W_2) \\
\downarrow r_{z_{12}; W_1} \boxtimes_{P(z_2)} 1_{W_2} & & \downarrow 1_{W_1} \boxtimes_{P(z_1)} l_{W_2} \\
W_1 \boxtimes_{P(z_2)} W_2 & \xrightarrow{\mathcal{T}_\gamma} & W_1 \boxtimes_{P(z_1)} W_2.
\end{array} \tag{12.90}$$

Let  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$ . Then we have

$$\begin{aligned}
& \overline{(1_{W_1} \boxtimes_{P(z_1)} l_{z_2; W_2}) \circ (\mathcal{A}_{P(z_1), P(z_2)}^{P(z_1 - z_2), P(z_2)})^{-1} ((w_{(1)} \boxtimes_{P(z_{12})} \mathbf{1}) \boxtimes_{P(z_2)} w_{(2)})} \\
&= \overline{(1_{W_1} \boxtimes_{P(z_1)} l_{z_2; W_2}) (w_{(1)} \boxtimes_{P(z_1)} (\mathbf{1} \boxtimes_{P(z_2)} w_{(2)}))} \\
&= w_{(1)} \boxtimes_{P(z_1)} w_{(2)}.
\end{aligned} \tag{12.91}$$

But

$$\begin{aligned}
& \overline{r_{z_{12}; W_1} \boxtimes_{P(z_2)} 1_{W_2} ((w_{(1)} \boxtimes_{P(z_{12})} \mathbf{1}) \boxtimes_{P(z_2)} w_{(2)})} \\
&= (e^{z_{12}L(-1)} w_{(1)}) \boxtimes_{P(z_2)} w_{(2)}.
\end{aligned} \tag{12.92}$$

Let  $\mathcal{Y} = \mathcal{Y}_{\boxtimes_{P(z_1)}, 0}$  be the intertwining operator of type  $\binom{W_1 \boxtimes_{P(z_1)} W_2}{W_1 \ W_2}$  corresponding to the  $P(z_1)$ -intertwining map  $\boxtimes_{P(z_1)}$ . Then by the definition of the parallel transport isomorphism and the  $L(-1)$ -derivative property for intertwining operators, we have

$$\begin{aligned}
\overline{\mathcal{T}_\gamma((e^{z_{12}L(-1)} w_{(1)}) \boxtimes_{P(z_2)} w_{(2)})} &= \mathcal{Y}(e^{z_{12}L(-1)} w_{(1)}, z_2) w_{(3)} \\
&= \mathcal{Y}(w_{(1)}, z_1) w_{(3)} \\
&= w_{(1)} \boxtimes_{P(z_1)} w_{(2)}.
\end{aligned} \tag{12.93}$$

Since the elements  $(w_{(1)} \boxtimes_{P(z_{12})} \mathbf{1}) \boxtimes_{P(z_2)} w_{(3)}$  for  $w_{(1)} \in W_1$  and  $w_{(2)} \in W_2$  span  $(W_1 \boxtimes_{P(z_{12})} V) \boxtimes_{P(z_2)} W_3$ , (12.91)–(12.93) give the commutativity of (12.90).

Let  $\gamma_1$  be a path from  $z_1$  to 1 in  $\mathbb{C}$  with a cut along the nonnegative real line. Let  $\gamma_2$  be the product of  $\gamma$  and  $\gamma_1$ . In particular,  $\gamma_2$  is a path from  $z_2$  to 1 in  $\mathbb{C}$  with a cut along the nonnegative real line. Also let  $\gamma_{12}$  be a path from  $z_{12} = z_1 - z_2$  to 1 in  $\mathbb{C}$  with a cut along the nonnegative real line. Then we have the following commutative diagrams:

$$\begin{array}{ccc} (W_1 \boxtimes V) \boxtimes W_2 & \xrightarrow{\mathcal{A}^{-1}} & W_1 \boxtimes (V \boxtimes W_2) \\ \downarrow \tau_{\gamma_2}^{-1} \circ (\tau_{\gamma_{12}}^{-1} \boxtimes_{P(z_2)} 1_{W_2}) & & \downarrow \tau_{\gamma_1}^{-1} \circ (1_{W_1} \boxtimes_{P(z_1)} \tau_{\gamma_2}^{-1}) \\ (W_1 \boxtimes_{P(z_{12})} V) \boxtimes_{P(z_2)} W_2 & \xrightarrow{(\mathcal{A}_{P(z_1), P(z_2)}^{P(z_{12}), P(z_2)})^{-1}} & W_1 \boxtimes_{P(z_1)} (V \boxtimes_{P(z_2)} W_2). \end{array} \quad (12.94)$$

$$\begin{array}{ccc} (W_1 \boxtimes V) \boxtimes W_2 & \xrightarrow{\tau_{\gamma_2}^{-1} \circ (\tau_{\gamma_{12}}^{-1} \boxtimes 1_{W_2})} & (W_1 \boxtimes_{P(z_{12})} V) \boxtimes_{P(z_2)} W_2 \\ \downarrow r_{W_1} \boxtimes 1_{W_2} & & \downarrow r_{W_1} \boxtimes_{P(z_2)} 1_{W_2} \\ W_1 \boxtimes W_2 & \xrightarrow{\tau_{\gamma_2}^{-1}} & W_1 \boxtimes_{P(z_2)} W_2. \end{array} \quad (12.95)$$

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_2) & \xrightarrow{\tau_{\gamma_1} \circ (1_{W_1} \boxtimes_{P(z_1)} \tau_{\gamma_2})} & W_1 \boxtimes (V \boxtimes W_2) \\ \downarrow 1_{W_1} \boxtimes_{P(z_1)} l_{W_2} & & \downarrow 1_{W_1} \boxtimes l_{W_2} \\ W_1 \boxtimes_{P(z_1)} W_2 & \xrightarrow{\tau_{\gamma_1}} & W_1 \boxtimes W_2. \end{array} \quad (12.96)$$

$$\begin{array}{ccc} W_1 \boxtimes_{P(z_2)} W_2 & \xrightarrow{\tau_{\gamma}} & W_1 \boxtimes_{P(z_1)} W_2 \\ \tau_{\gamma_2} \downarrow & & \downarrow \tau_{\gamma_1} \\ W_1 \boxtimes W_2 & = & W_1 \boxtimes W_2. \end{array} \quad (12.97)$$

The commutativity of (12.94) follows from the definition of  $\mathcal{A}$ . The commutativity of (12.95) and (12.96) follows from the definition of the left and right unit isomorphisms and the parallel transport isomorphisms. The commutativity of (12.97) follows from the fact that  $\gamma_2$  is the product of  $\gamma$  and  $\gamma_1$ . Combining (12.90) and (12.94)–(12.97), we obtain the commutativity of the triangle diagram for the unit isomorphisms.

It is clear from the definition that  $l_V = r_V$ .

Thus we have proved that the category  $\mathcal{C}$  equipped with the data given in Section 12.2 is a braided monoidal category.  $\square$

In the case that  $\mathcal{C}$  is an abelian category, we have:

**Corollary 12.16** *If the category  $\mathcal{C}$  is an abelian category, then  $\mathcal{C}$ , equipped with the tensor product bifunctor  $\boxtimes$ , the unit object  $V$ , the braiding isomorphism  $\mathcal{R}$ , the associativity isomorphism  $\mathcal{A}$ , and the left and right unit isomorphisms  $l$  and  $r$ , is a braided tensor category.*

$\square$

**Remark 12.17** As we mentioned at the beginning of this section, this braided tensor category structure has “forgotten” the underlying complex-analytic vertex-tensor-categorical structure that has in fact been developed in this work, retaining only its “topological” part, but our proof has needed the vertex-tensor-categorical structure, essentially because iterated tensor products of triples of elements are not defined in the braided tensor category structure (recall Remark 12.11). Also, our category has a contragredient functor (which we have been using extensively), although in the present generality, we do not have rigidity.

## References

- [BK] B. Bakalov and A. Kirillov, Jr., *Lectures on tensor categories and modular functors*, University Lecture Series, Vol. 21, Amer. Math. Soc., Providence, RI, 2001.
- [H1] Y.-Z. Huang, A theory of tensor products for module categories for a vertex operator algebra, IV, *J. Pure Appl. Alg.* **100** (1995) 173–216.
- [H2] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Comm. Contemp. Math.* **10** (2008), 871–911.
- [HL1] Y.-Z. Huang and J. Lepowsky, Tensor products of modules for a vertex operator algebras and vertex tensor categories, in: *Lie Theory and Geometry, in honor of Bertram Kostant*, ed. R. Brylinski, J.-L. Brylinski, V. Guillemin, V. Kac, Birkhäuser, Boston, 1994, 349–383.
- [HL2] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, I, *Selecta Mathematica (New Series)* **1** (1995), 699–756.
- [HL3] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, II, *Selecta Mathematica (New Series)* **1** (1995), 757–786.
- [HL4] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, III, *J. Pure Appl. Alg.* **100** (1995) 141–171.
- [HL5] Y.-Z. Huang and J. Lepowsky, A theory of tensor products for module categories for a vertex operator algebra, V, to appear.
- [HLZ1] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, to appear.
- [HLZ2] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, II: Logarithmic formal calculus and properties of logarithmic intertwining operators, to appear.

- [HLZ3] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, III: Intertwining maps and tensor product bifunctors, to appear.
- [HLZ4] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, IV: Constructions of tensor product bifunctors and the compatibility conditions, to appear.
- [HLZ5] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, V: Convergence condition for intertwining maps and the corresponding compatibility condition, to appear.
- [HLZ6] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms, to appear.
- [HLZ7] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps, to appear.
- [Kn] A. W. Knap, *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton, New Jersey, 1986.
- [M] S. Mac Lane, *Categories for the Working Mathematicians*, Graduate Text in Mathematics, Vol. 5, Springer-Verlag, New York, 1971.
- [T] V. G. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Math., Vol. 18, Walter de Gruyter, Berlin, 1994.

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